Int J Theor Phys (2009) 48: 1973–1999
DOI 10.1007/s10773-009-9973-5

Einstein Gravity in Almost Kähler Variables and Stability of Gravity with Nonholonomic Distributions and Nonsymmetric Metrics

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Received: 7 November 2008 / Accepted: 25 February 2009 / Published online: 11 March 2009

Abstract We argue that the Einstein gravity theory can be reformulated in almost Kähler (nonsymmetric) variables with effective symplectic form and compatible linear connection uniquely defined by a (pseudo) Riemannian metric. A class of nonsymmetric theories of gravitation on manifolds enabled with nonholonomic distributions is considered. We prove that, for certain types of nonholonomic constraints, there are modelled effective Lagrangians which do not develop instabilities. It is also elaborated a linearization formalism for anholonomic noncommutative gravity theories models and analyzed the stability of stationary ellipsoidal solutions defining some nonholonomic and/or nonsymmetric deformations of the Schwarzschild metric. We show how to construct nonholonomic distributions which remove instabilities in nonsymmetric gravity theories. It is concluded that instabilities do not consist a general feature of theories of gravity with nonsymmetric metrics but a particular property of some models and/or unconstrained solutions.

Keywords Gravity and symplectic variables · Nonsymmetric metrics · Nonholonomic manifolds · Nonlinear connections · Stability

1 Introduction

In this article, we re-address the issue of nonsymmetric gravity theory following three key ideas: 1) the general relativity theory can be written equivalently in terms of certain nonsymmetric variables; 2) nonsymmetric contributions to metrics and connections may be generated in quasi-classical limits of quantum gravity and nonholonomic and/or noncommutative Ricci flow theory; 3) physically valuable solutions and their generalizations with

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nonsymmetric/noncommutative/nonholonomic variables can be stabilized by corresponding classes of nonholonomic constraints on gravitational field and (geometric) evolution equations. This paper belongs to a series of three our works on gravity and spaces enabled with general symmetric nonsymmetric components metrics and related nonlinear and linear connection structures, see also partner articles [1, 2]. Our goal is to consider some knew applications in gravity physics and define the conditions when such gravitational "nonsymetric" interactions can be modelled on Einstein spaces.

In our recent papers on quantum gravity [3, 4], we worked with almost Kähler (canonical almost symplectic) variables, see a review of results and applications of the geometric formalism for constructing exact solutions in gravity [5] and modelling locally anisotropic interactions in standard theories of physics [6]. In this article, we shall construct such symplectic (nonsymmetric) nonholonomic variables for classical Einstein gravity and possible generalizations to nonsymmetric gravity theories. The almost symplectic/Kähler connection in gravity is similar to the Cartan connection in Finsler–Lagrange geometry [7–14], but we emphasize that in this article we shall work only with geometric structures defined on nonholonomic (pseudo) Riemannian manifolds.[1]

From a formal point of view, the Cartan's almost symplectic connection contains nontrivial torsion components induced by the anholonomy coefficients.[2] Such a nonholonomically induced torsion is not similar to torsions from the Einstein–Cartan and/or string/gauge gravity theories, where certain additional field equations (to the Einstein equations) are considered for torsion fields.

For the almost Kähler representation of general relativity, the gravitational symplectic form is anti-symmetric, $\theta_{\mu\nu} = -\theta_{\nu\mu}$, and play the role of "anti-symmetric" metric. We can consider additional "nonsymmetric" metric contributions from "de-quantization" procedure in deformation quantization of gravity, or (in a more straightforward form) from the theory of nonholonomic and/or noncommutative Ricci flows, see [2, 15, 16]. Such geometric quantum constructions and evolution models put in a new fashion the problem of gravity with nonsymmetric variables. There is already a long time history, beginning with A. Einstein [17, 18] and L.P. Eisenhart [19, 20], when the so-called nonsymmetric gravity theories have been elaborated in different modifications by J. Moffat and co-authors [21–27], see also a recent paper [28]. Here we note that a more general class of geometries with nonsymmetric metrics and nonlinear and linear connections, generalizing the concept of Lagrange and Finsler spaces was investigated in [29, 30].

A series of works by T. Janssen and T. Prokopec [31–33] is devoted to the so-called "problem of instabilities" in nonsymmetric gravity theories. The authors agreed that one can be elaborated such models with nonzero mass term for the nonsymmetric part of metric (treated as an absolutely symmetric torsion induced by an effective B-field like in string gravity, but in four dimensions). That solved the problems formally created by absence of gauge invariance found by Damour, Deser and McCarthy [34], see explicit constructions and detailed discussions in [23, 35]. It was also emphasized that, as a matter of principle, the

[1] A pair $(\mathbf{V}, \mathcal{N})$, where $\mathbf{V}$ is a manifold and $\mathcal{N}$ is a nonintegrable distribution on $\mathbf{V}$, is called a nonholonomic manifold; we note that in our works we use left "up" and "low" symbols as formal labels for certain geometric objects and that the spacetime signature may be encoded into formal frame (vielbein) coefficients, some of them being proportional to the imaginary unity i, when $i^2 = -1$.

[2] In mathematical and physical literature, there are used also some other equivalent terms like anholonomic, or non-integrable, restrictions/constraints; we emphasize that in classical and quantum physics the field and evolution equations play a fundamental role but together with certain types of constraints and broken symmetries.

Clayton's effect [36] (when, for a general relativity background, a small B-field for the nonsymmetric part quickly grows) may be stabilized by solutions with evolving backgrounds [37] and/or introducing an extra Lagrange multiplier when the unstable modes dynamically vanish [38].

Nevertheless, the general conclusion following from works [31–33] is that instabilities in nonsymmetric gravity theory should not be seen as a relict of the linearized theory because certain nonlinearized nonsymmetric gravity models with nontrivial Einstein background (for instance, on Schwarzschild spacetime) are positively unstable. Such solutions can not be stabilized by the former methods with dynamical solutions and, as a consequence, certain new models of nonsymmetric gravity models and methods of stabilizations should be developed.

It should be emphasized that the Janssen–Prokopek stability problem does not have a generic character for all models of gravity with nonsymmetric variables. As we emphasized above, the Einstein gravity can be represented equivalently in canonical almost symplectic variables and such a formal theory with nonsymmetric metric (nonholonomically transformed into components of a symplectic form) is stable under deformations of the Schwarzschild metric. But in such a representation, we have also certain nontrivial nonholonomic structures. So, it is important to study the problem of stability of physical valuable solutions in general relativity under nonholonomic deformations, which may keep the constructions in the framework of the Einstein theory (with certain classes of imposed non-integrable constraints), or may generalize the gravity theory to models with nontrivial contributions from Ricci flow evolution (for instance, under variation of gravitational constants) and/or from a noncommutative/quantum gravity theory.

The goal of this paper is to prove that stable configurations can be derived for various models of nonsymmetric gravity theories [21–27]. We shall use a geometric techniques elaborated in [1, 2, 6, 39, 40] and show how nonholonomic frame constraints can be imposed in order to generate stable solutions in nonsymmetric gravity theories. For vanishing nonsymmetric components of metrics such configurations can be reduced to nonholonomic[3] ones in general relativity theory and generalizations. We shall provide explicit examples of stationary solutions with ellipsoidal symmetry which can be constructed in nonsymmetric gravity and general relativity theories; such metrics are stable and transform into the Schwarzschild one for zero eccentricities.

In brief, the Janssen–Prokopec method proving that a full, nonlinearized, nonsymmetric gravity theories may suffer from instabilities can be summarized in this form: One shows that there is only one stable linearized Lagrangian (see in [31] the formula (A26) which can be obtained from their formula (86); similar formulas, (43) and (45), are provided below in Sect. 3). Then, following certain explicit computations for different backgrounds in general relativity, one argues that for the Schwarzschild background the mentioned variant of stable Lagrangian cannot be obtained by linearizing nonsymmetric gravity theories (because in such cases, the coefficient γ in the mentioned formulas, can not be zero for the static spherical symmetric background in general relativity).

Generalizing the constructions from [31] in order to include certain types of nonholonomic distributions on (non) symmetric spacetime manifolds, we shall prove that stable Lagrangians can be generated by a superposition of nonholonomic transforms and linearization in general models of nonsymmetric gravity theories with compatible (nonsymmetric) metrics and nonlinear and linear connection structures.

[3] Equivalently, there are used the terms anholonomic and/or nonintegrable.

We argue that fixing from the very beginning an ansatz with spherical symmetry background (for instance, the Schwarzschild solution in gravity), one eliminates from consideration a large class of physically important symmetric and nonsymmetric nonlinear gravitational interactions. The resulting instability of such constrained to a given background solutions reflects the proprieties of some very special classes of solutions but not any intrinsic, fundamental, general characteristics of nonsymmetric gravity theories. For instance, we shall construct explicit "ellipsoidal" stationary solutions in nonsymmetric gravity theories to which a static Schwarzschild metric is deformed by very small nonsymmetric metric components and nonholonomic distributions and which seem to be stable for geometric distorsions in Einstein gravity [41, 42].[4] Such metrics were constructed for different models of metric-affine, generalized Finsler on nonholonomic manifolds and noncommutative gravity [5, 40, 43]) and can be included in nonsymmetric gravity theories both by nonsymmetric metric components and/or as a nonholonomic symmetric background, see examples from [2].

One should be noted that in a number of works on gravity with nonsymmetric metrics the short term NGT is used [35, 36] (instead of nonsymmetric gravity theory/-ies, see details in [22–24, 38]). This may result in some misunderstanding with the "noncommutative gravity theory" developed in some approaches to noncommutative geometry and applications. In order to avoid ambiguities with the term NGT, in this work, we shall write explicitly the words "nonsymmetric gravity theory/-ies", for geometric and physical models with nonsymmetric metrics on commutative (in general, nonholonomic) manifolds. We shall not consider possible relations between nonsymmetric metrics and noncommutative geometry. Here we also emphasize that black ellipsoid solutions analyzed in Sect. 4 of this work are only for nonholonomic deformations in general relativity and nonsymmetric gravity theories. Similar classes of solutions have been constructed in [41–43] (see also Parts I and II in monograph [40], for (non) commutative metric-affine, gauge and string gravity generalizations) and positively have certain connections solutions defining "noncommutative black holes" [44, 45].

The paper is organized as follows: In Sect. 2, we outline some basic results from the geometry of nonholonomic manifolds and nonsymmetric gravity models on such spaces. The equivalent formulation of the Einstein gravity in canonical almost symplectic variables is provided. Section 3 is devoted to a method of nonholonomic deformations and linearization to backgrounds with symmetric metrics and nonholonomic distributions. We show how certain classes of nonsymmetric metric configurations can be stabilized by corresponding nonholonomic constraints. We present an explicit example in Sect. 4, when stable stationary solutions with nontrivial nonsymmetric components of metric and nonholonomic distributions are constructed as certain deformations of the Schwarzschild metric to an ellipsoidal nonholonomic background on which a constrained dynamics on nonsymmetric metric fields is modelled. Finally, in Sect. 5 we present conclusions and discuss the results. In Appendix, we provide some important formulas on torsion and curvature of linear connections adapted to a prescribed nonlinear connection structure.

2 Almost Kähler Variables in Einstein and Nonsymmetric Gravity Theories

In general relativity, we consider a real four dimensional (pseudo) Riemannian spacetime manifold V of signature $(-,+,+,+)$ and necessary smooth class. For a conventional $2+2$

[4] In this work, we can consider that the nonsymmetric components of a general metric induce such geometric and effective matter field distortions.

splitting, the local coordinates $u=(x,y)$ on a open region $U\subset V$ are labelled in the form $u^\alpha=(x^i,y^a)$, where indices of type $i,j,k,\ldots=1,2$ and $a,b,c\ldots=3,4$, for tensor like objects, will be considered with respect to a general (non-coordinate) local basis $e_\alpha=(e_i,e_a)$. One says that x^i and y^a are respectively the conventional horizontal/nonholonomic (h) and vertical/holonomic (v) coordinates (both types of such coordinates can be time- or space-like ones). Primed indices of type $i',a',\ldots$ will be used for labelling coordinates with respect to a different local basis $e_{\alpha'}=(e_{i'},e_{a'})$ or $e_{\alpha'}=(e'_0,e_{I'})$, for instance, for an orthonormalized basis. For the local tangent Minkowski space, we chose $e_{0'}=i\partial/\partial u^{0'}$, where i is the imaginary unity, $i^2=-1$, and write $e_{\alpha'}=(i\partial/\partial u^{0'},\partial/\partial u^{1'},\partial/\partial u^{2'},\partial/\partial u^{3'})$. To consider such formal Euclidean coordinates is useful for some purposes of analogous modelling of gravity theories as effective Lagrange mechanics geometries, but this does not mean that we introduce any complexification of classical spacetimes. In this section, we outline the constructions for classical gravity from [3, 4, 12].

2.1 N-Anholonomic (Pseudo) Riemannian Manifolds

The coefficients of a general (pseudo) Riemannian metric on a spacetime V are parametrized in the form:

$$\begin{aligned}\mathbf{g}&=g_{i'j'}(u)e^{i'}\otimes e^{j'}+h_{a'b'}(u)e^{a'}\otimes e^{b'},\\ e^{a'}&=\mathbf{e}^{a'}-N_{i'}^{a'}(u)e^{i'},\end{aligned}\tag{1}$$

where the required form of vierbein coefficients $e_\alpha^{\alpha'}$ of the dual basis $e^{\alpha'}=(e^{i'},e^{a'})=e_\alpha^{\alpha'}(u)du^\alpha$, defining a formal $2+2$ splitting, will be stated below.

On spacetime V, we consider any generating function $L(u)=L(x^i,y^a)$ (we may call it a formal Lagrangian if an effective continuous mechanical model of general relativity is to be elaborated, see [6, 39, 40]) with nondegenerate Hessian

$${}^Lh_{ab}=\frac{1}{2}\frac{\partial^2L}{\partial y^a\partial y^b},\tag{2}$$

when $\det|{}^Lh_{ab}|\neq 0$. This function is useful for constructing in explicit form a nonholonomic $2+2$ splitting for which a canonical almost symplectic model of general relativity will be defined. We use L as an abstract label and emphasize that the geometric constructions are general ones, not depending on the type of function $L(u)$ which states only a formal class of systems of reference and coordinates. We introduce

$${}^LN_i^a=\frac{\partial G^a}{\partial y^{2+i}},\tag{3}$$

for

$$G^a=\frac{1}{4}\,{}^Lh^{a\ 2+i}\left(\frac{\partial^2L}{\partial y^{2+i}\partial x^k}y^{2+k}-\frac{\partial L}{\partial x^i}\right),\tag{4}$$

where ${}^Lh^{ab}$ is inverse to ${}^Lh_{ab}$ and respective contractions of h- and v-indices, $i,j,\ldots$ and $a,b\ldots$, are performed following the rule: we can write, for instance, an up v-index a as $a=2+i$ and contract it with a low index $i=1,2$. Briefly, we shall write y^i instead of y^{2+i}, or y^a. The values (2), (3) and (4) allow us to define

$$
\begin{aligned}
{}^L\mathbf{g} &= {}^L g_{ij} dx^i \otimes dx^j + {}^L h_{ab}\, {}^L\mathbf{e}^a \otimes {}^L\mathbf{e}^b, \\
{}^L\mathbf{e}^a &= dy^a + {}^L N_i^a dx^i, \qquad {}^L g_{ij} = {}^L h_{2+i\ 2+j}.
\end{aligned} \tag{5}
$$

A metric $\mathbf{g}$ (1) with coefficients $g_{\alpha'\beta'} = [g_{i'j'}, h_{a'b'}]$ computed with respect to a dual basis $e^{\alpha'} = (e^{i'}, e^{a'})$ can be related to the metric ${}^L\mathbf{g}_{\alpha\beta} = [{}^L g_{ij},\ {}^L h_{ab}]$ (5) with coefficients defined with respect to a N-adapted dual basis ${}^L e^{\alpha} = (dx^i,\ {}^L\mathbf{e}^a)$ if there are satisfied the conditions

$$
\mathbf{g}_{\alpha'\beta'} e^{\alpha'}_{\ \alpha} e^{\beta'}_{\ \beta} = {}^L\mathbf{g}_{\alpha\beta}. \tag{6}
$$

Considering any given values $\mathbf{g}_{\alpha'\beta'}$ and ${}^L\mathbf{g}_{\alpha\beta}$, we have to solve a system of quadratic algebraic equations with unknown variables $e^{\alpha'}_{\ \alpha}$. How to define locally such coordinates, we discuss in [6, 14]. For instance, in general relativity, there are 6 independent values $\mathbf{g}_{\alpha'\beta'}$ and up till ten coefficients ${}^L\mathbf{g}_{\alpha\beta}$ which allows us always to define a set of vierbein coefficients $e^{\alpha'}_{\ \alpha}$. Usually, a subset of such coefficients can be taken to be zero, for given values $[g_{i'j'}, h_{a'b'}, N^{a'}_{i'}]$ and $[{}^L g_{ij},\ {}^L h_{ab},\ {}^L N^a_i]$, when

$$
N^{a'}_{i'} = e_{i'}^{\ i} e^{a'}_{\ a}\ {}^L N^a_i \tag{7}
$$

for $e_{i'}^{\ i}$ being inverse to $e^{i'}_{\ i}$.

For simplicity, in this work, we suppose that there is always a finite covering of $\mathbf{V}^{2+2}$ (in brief, denoted $\mathbf{V}$) by a family of open regions ${}^I U$, labelled by an index I, on which there are considered certain nontrivial effective Lagrangians ${}^I L$ with real solutions ${}^I e^{\alpha'}_{\ \alpha}$ defining vielbein transforms to systems of so-called Lagrange variables. Finally, we solve the algebraic equations (6) for any prescribed values $g_{i'j'}$ (we also have to change the partition ${}^I U$ and generating function ${}^I L$ till we are able to construct real solutions) and find ${}^I e^{i'}_{\ i}$ which, in its turn, allows us to compute $N^{a'}_{i'}$ (7) and all coefficients of the metric $\mathbf{g}$ (1) and vierbein transform. We shall omit for simplicity the left label L if that will not result in a confusion for some special constructions.

A nonlinear connection (N-connection) structure $\mathbf{N}$ for $\mathbf{V}$ is defined by a nonholonomic distribution (a Whitney sum)

$$
T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V} \tag{8}
$$

into conventional horizontal (h) and vertical (v) subspaces. In local form, a N-connection is given by its coefficients $N^a_i(u)$, when

$$
\mathbf{N} = N^a_i(u) dx^i \otimes \frac{\partial}{\partial y^a}. \tag{9}
$$

A N-connection introduces on $\mathbf{V}^{n+n}$ a frame (vielbein) structure

$$
\mathbf{e}_\nu = \left(\mathbf{e}_i = \frac{\partial}{\partial x^i} - N^a_i(u)\frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a}\right), \tag{10}
$$

$$
\mathbf{e}^\mu = \left(e^i = dx^i, \mathbf{e}^a = dy^a + N^a_i(u) dx^i\right). \tag{11}
$$

The vielbeins (11) satisfy the nonholonomy relations

$$
[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = w^\gamma_{\alpha\beta} \mathbf{e}_\gamma \tag{12}
$$

with (antisymmetric) nontrivial anholonomy coefficients $w^b_{ia} = \partial_a N^b_i$ and $w^a_{ji} = \Omega^a_{ij}$, where

$$\Omega^a_{ij} = \mathbf{e}_j\left(N^a_i\right) - \mathbf{e}_i\left(N^a_j\right) \tag{13}$$

define the coefficients of N-connection curvature. The particular holonomic/integrable case is selected by the integrability conditions $w^\gamma_{\alpha\beta} = 0$.[5]

A N-anholonomic manifold is a (nonholonomic) manifold enabled with N-connection structure (8). The geometric properties of a N-anholonomic manifold are distinguished by some N-adapted bases (10) and (11). A geometric object is N-adapted (equivalently, distinguished), i.e. a d-object, if it can be defined by components adapted to the splitting (8) (one uses terms d-vector, d-form, d-tensor). For instance, a d-vector $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a$ and a one d-form $\widetilde{\mathbf{X}}$ (dual to $\mathbf{X}$) is $\widetilde{\mathbf{X}} = X_\alpha \mathbf{e}^\alpha = X_i e^i + X_a e^a$.[6]

2.2 Canonical Almost Symplectic Structures in General Relativity

Let $\mathbf{e}_{\alpha'} = (\mathbf{e}_i, e_{b'})$ and $\mathbf{e}^{\alpha'} = (e^i, \mathbf{e}^{b'})$ be defined respectively by (10) and (11) for the canonical N-connection ${}^L\mathbf{N}$ (3) stated by a metric structure $\mathbf{g} = {}^L\mathbf{g}$ (5) on $\mathbf{V}$. We introduce a linear operator $\mathbf{J}$ acting on vectors on $\mathbf{V}$ following formulas $\mathbf{J}(\mathbf{e}_i) = -e_{2+i}$ where $\mathbf{J}(e_{2+i}) = \mathbf{e}_i$, where $\mathbf{J} \circ \mathbf{J} = -\mathbf{I}$, for $\mathbf{I}$ being the unity matrix. Alternatively, $\mathbf{J}$ can be regarded as a tensor field on $\mathbf{V}$,

$$\begin{aligned}
\mathbf{J} &= \mathbf{J}^\alpha_{\ \beta}\, e_\alpha \otimes e^\beta = \mathbf{J}^{\underline{\alpha}}_{\ \underline{\beta}} \frac{\partial}{\partial u^{\underline{\alpha}}} \otimes du^{\underline{\beta}} \\
&= \mathbf{J}^{\alpha'}_{\ \beta'}\, \mathbf{e}_{\alpha'} \otimes \mathbf{e}^{\beta'} = -e_{2+i} \otimes e^i + \mathbf{e}_i \otimes \mathbf{e}^{2+i} \\
&= -\frac{\partial}{\partial y^i} \otimes dx^i + \left(\frac{\partial}{\partial x^i} - {}^L N^{2+j}_i \frac{\partial}{\partial y^j}\right) \otimes \left(dy^i + {}^L N^{2+i}_k dx^k\right),
\end{aligned} \tag{14}$$

defining globally an almost complex structure on $\mathbf{V}$ completely determined by a fixed $L(x,y)$. Using vielbeins $\mathbf{e}^{\alpha}_{\ \underline{\alpha}}$ and their duals $\mathbf{e}_{\alpha}^{\ \underline{\alpha}}$, defined by $e^{\alpha'}_{\ \alpha}$ solving (6), we can compute the coefficients of tensor $\mathbf{J}$ with respect to any local basis e_α and e^α on $\mathbf{V}$, $\mathbf{J}^\alpha_{\ \beta} = \mathbf{e}^\alpha_{\ \underline{\alpha}} \mathbf{J}^{\underline{\alpha}}_{\ \underline{\beta}} \mathbf{e}_\beta^{\ \underline{\beta}}$. In general, we can define an almost complex structure $\mathbf{J}$ for an arbitrary N-connection $\mathbf{N}$, stating a nonholonomic $2+2$ splitting, by using N-adapted bases (10) and (11).

The Neijenhuis tensor field for any almost complex structure $\mathbf{J}$ defined by a N-connection (equivalently, the curvature of N-connection) is

$$^{\mathbf{J}}\Omega(\mathbf{X},\mathbf{Y}) \doteqdot -[\mathbf{X},\mathbf{Y}] + [\mathbf{JX},\mathbf{JY}] - \mathbf{J}[\mathbf{JX},\mathbf{Y}] - \mathbf{J}[\mathbf{X},\mathbf{JY}], \tag{15}$$

for any d-vectors $\mathbf{X}$ and $\mathbf{Y}$. With respect to N-adapted bases (10) and (11), a subset of the coefficients of the Neijenhuis tensor defines the N-connection curvature, see details in [11],

$$\Omega^a_{ij} = \frac{\partial N^a_i}{\partial x^j} - \frac{\partial N^a_j}{\partial x^i} + N^b_i \frac{\partial N^a_j}{\partial y^b} - N^b_j \frac{\partial N^a_i}{\partial y^b}. \tag{16}$$

[5] We use boldface symbols for spaces (and geometric objects on such spaces) enabled with N-connection structure.

[6] We can redefine equivalently the geometric constructions for arbitrary frame and coordinate systems; the N-adapted constructions allow us to preserve the h- and v-splitting.

A N-anholonomic manifold $\mathbf{V}$ is integrable if $\Omega^a_{ij}=0$. We get a complex structure if and only if both the h- and v-distributions are integrable, i.e. if and only if $\Omega^a_{ij}=0$ and $\frac{\partial N^a_j}{\partial y^i}-\frac{\partial N^a_i}{\partial y^j}=0$.

One calls an almost symplectic structure on a manifold V a nondegenerate 2-form $\theta=\frac{1}{2}\theta_{\alpha\beta}(u)e^\alpha\wedge e^\beta$. There is a unique N-connection $\mathbf{N}=\{N^a_i\}$ (8) satisfying the conditions:

$$\theta(h\mathbf{X},v\mathbf{Y})=0 \quad \text{and} \quad \theta=h\theta+v\theta, \tag{17}$$

for $\mathbf{X}=h\mathbf{X}+v\mathbf{X}$, $\mathbf{Y}=h\mathbf{Y}+v\mathbf{Y}$, where $h\theta(\mathbf{X},\mathbf{Y})\doteqdot\theta(h\mathbf{X},h\mathbf{Y})$ and $v\theta(\mathbf{X},\mathbf{Y})\doteqdot\theta(v\mathbf{X},v\mathbf{Y})$.

For $\mathbf{X}=\mathbf{e}_\alpha=(\mathbf{e}_i,e_a)$ and $\mathbf{Y}=\mathbf{e}_\beta=(\mathbf{e}_l,e_b)$, where $\mathbf{e}_\alpha$ is a N-adapted basis of type (10), we write the first equation in (17) in the form $\theta(\mathbf{e}_i,e_a)=\theta(\frac{\partial}{\partial x^i},\frac{\partial}{\partial y^a})-N^b_i\theta(\frac{\partial}{\partial y^b},\frac{\partial}{\partial y^a})=0$. We can solve this system of equations in a unique form and define N^b_i if $\mathrm{rank}|\theta(\frac{\partial}{\partial y^b},\frac{\partial}{\partial y^a})|=2$. Denoting locally

$$\theta=\frac{1}{2}\theta_{ij}(u)e^i\wedge e^j+\frac{1}{2}\theta_{ab}(u)\mathbf{e}^a\wedge\mathbf{e}^b, \tag{18}$$

where the first term is for $h\theta$ and the second term is $v\theta$, we get the second formula in (17).

An almost Hermitian model of a (pseudo) Riemannian space $\mathbf{V}$ equipped with a N-connection structure $\mathbf{N}$ is defined by a triple $\mathbf{H}^{2+2}=(\mathbf{V},\theta,\mathbf{J})$, where $\theta(\mathbf{X},\mathbf{Y})\doteqdot\mathbf{g}(\mathbf{J}\mathbf{X},\mathbf{Y})$ for any $\mathbf{g}$ (1). A space $\mathbf{H}^{2+2}$ is almost Kähler, denoted $\mathbf{K}^{2+2}$, if and only if $d\theta=0$.

For $\mathbf{g}={}^L\mathbf{g}$ (5) and structures ${}^L\mathbf{N}$ (3) and $\mathbf{J}$ canonically defined by L, we define ${}^L\theta(\mathbf{X},\mathbf{Y})\doteqdot{}^L\mathbf{g}(\mathbf{J}\mathbf{X},\mathbf{Y})$ for any d-vectors $\mathbf{X}$ and $\mathbf{Y}$. In local N-adapted form, we have

$$\begin{aligned}{}^L\theta&=\frac{1}{2}\,{}^L\theta_{\alpha\beta}(u)e^\alpha\wedge e^\beta=\frac{1}{2}\,{}^L\theta_{\underline{\alpha}\underline{\beta}}(u)du^{\underline{\alpha}}\wedge du^{\underline{\beta}}\\&={}^Lg_{ij}(x,y)e^{2+i}\wedge dx^j={}^Lg_{ij}(x,y)(dy^{2+i}+{}^LN^{2+i}_k dx^k)\wedge dx^j.\end{aligned} \tag{19}$$

Let us consider the form ${}^L\omega=\frac{1}{2}\frac{\partial L}{\partial y^i}dx^i$. A straightforward computation shows that ${}^L\theta=d\,{}^L\omega$, which means that $d\,{}^L\theta=dd\,{}^L\omega=0$, i.e. the canonical effective Lagrange structures $\mathbf{g}={}^L\mathbf{g}$, ${}^L\mathbf{N}$ and $\mathbf{J}$ induce an almost Kähler geometry. We can express the 2-form (19) as

$$\begin{aligned}\theta={}^L\theta&=\frac{1}{2}\,{}^L\theta_{ij}(u)e^i\wedge e^j+\frac{1}{2}\,{}^L\theta_{ab}(u)\mathbf{e}^a\wedge\mathbf{e}^b\\&=g_{ij}(x,y)\left[dy^i+N^i_k(x,y)dx^k\right]\wedge dx^j,\end{aligned} \tag{20}$$

see (18), where the coefficients ${}^L\theta_{ab}={}^L\theta_{2+i\ 2+j}$ are equal respectively to the coefficients ${}^L\theta_{ij}$. It should be noted that for a general 2-form θ constructed for any metric $\mathbf{g}$ and almost complex $\mathbf{J}$ structures on V one holds $d\theta\neq0$. But for any $2+2$ splitting induced by an effective Lagrange generating function, we have $d\,{}^L\theta=0$. We have also $d\theta=0$ for any set of 2-form coefficients $\theta_{\alpha'\beta'}e^{\alpha'}_{\ \alpha}e^{\beta'}_{\ \beta}={}^L\theta_{\alpha'\beta'}$ (such a 2-form θ will be called to be a canonical one), constructed by using formulas (6).

We conclude that having chosen a generating function $L(x,y)$ on a (pseudo) Riemannian spacetime $\mathbf{V}$, we can model this spacetime equivalently as an almost Kähler manifold

2.3 Equivalent Metric Compatible Linear Connections

A distinguished connection (in brief, d-connection) on a spacetime $\mathbf{V}$,

$$\mathbf{D}=(hD;vD)=\{\Gamma^\alpha_{\beta\gamma}=(L^i_{jk},\,{}^vL^a_{bk};C^i_{jc},\,{}^vC^a_{bc})\},$$

is a linear connection which preserves under parallel transports the distribution (8). In explicit form, the coefficients $\Gamma^{\alpha}_{\beta\gamma}$ are computed with respect to a N-adapted basis (10) and (11). A d-connection $\mathbf{D}$ is metric compatible with a d-metric $\mathbf{g}$ if $\mathbf{D}_{\mathbf{X}}\mathbf{g} = 0$ for any d-vector field $\mathbf{X}$.

If an almost symplectic structure θ is considered on a N-anholonomic manifold, an almost symplectic d-connection ${}_{\theta}\mathbf{D}$ on $\mathbf{V}$ is defined by the conditions that it is N-adapted, i.e. it is a d-connection, and ${}_{\theta}\mathbf{D}_{\mathbf{X}}\theta = 0$, for any d-vector $\mathbf{X}$. From the set of metric and/or almost symplectic compatible d-connections on a (pseudo) Riemannian manifold $\mathbf{V}$, we can select those which are completely defined by a metric $\mathbf{g} = {}^{L}\mathbf{g}$ (5) and an effective Lagrange structure $L(x, y)$:

There is a unique normal d-connection

$$\begin{aligned}\widehat{\mathbf{D}} &= \left\{h\widehat{D} = (\widehat{D}_k, {}^{v}\widehat{D}_k = \widehat{D}_k); v\widehat{D} = (\widehat{D}_c, {}^{v}\widehat{D}_c = \widehat{D}_c)\right\} \\ &= \{\widehat{\Gamma}^{\alpha}_{\beta\gamma} = (\widehat{L}^{i}_{jk}, {}^{v}\widehat{L}^{2+i}_{2+j\ 2+k} = \widehat{L}^{i}_{jk};\ \widehat{C}^{i}_{jc} = {}^{v}\widehat{C}^{2+i}_{2+j\ c}, {}^{v}\widehat{C}^{a}_{bc} = \widehat{C}^{a}_{bc})\},\end{aligned} \tag{21}$$

which is metric compatible, $\widehat{D}_k\, {}^{L}g_{ij} = 0$ and $\widehat{D}_c\, {}^{L}g_{ij} = 0$, and completely defined by a couple of h- and v-components $\widehat{\mathbf{D}}_{\alpha} = (\widehat{D}_k, \widehat{D}_c)$, with N-adapted coefficients $\widehat{\Gamma}^{\alpha}_{\beta\gamma} = (\widehat{L}^{i}_{jk}, {}^{v}\widehat{C}^{a}_{bc})$, where

$$\begin{aligned}\widehat{L}^{i}_{jk} &= \frac{1}{2}\, {}^{L}g^{ih}\left(\mathbf{e}_k\, {}^{L}g_{jh} + \mathbf{e}_j\, {}^{L}g_{hk} - \mathbf{e}_h\, {}^{L}g_{jk}\right), \\ \widehat{C}^{i}_{jk} &= \frac{1}{2}\, {}^{L}g^{ih}\left(\frac{\partial\, {}^{L}g_{jh}}{\partial y^k} + \frac{\partial\, {}^{L}g_{hk}}{\partial y^j} - \frac{\partial\, {}^{L}g_{jk}}{\partial y^h}\right).\end{aligned} \tag{22}$$

In general, we can "forget" about label L and work with arbitrary $\mathbf{g}_{\alpha'\beta'}$ and $\widehat{\Gamma}^{\alpha'}_{\beta'\gamma'}$ with the coefficients recomputed by frame transforms.

Introducing the normal d-connection 1-form $\widehat{\Gamma}^{i}_{j} = \widehat{L}^{i}_{jk}e^k + \widehat{C}^{i}_{jk}\mathbf{e}^k$, we prove that the Cartan structure equations are satisfied,

$$de^k - e^j \wedge \widehat{\Gamma}^{k}_{j} = -\widehat{\mathcal{T}}^{i}, \qquad d\mathbf{e}^k - \mathbf{e}^j \wedge \widehat{\Gamma}^{k}_{j} = -\,{}^{v}\widehat{\mathcal{T}}^{i}, \tag{23}$$

$$d\widehat{\Gamma}^{i}_{j} - \widehat{\Gamma}^{h}_{j} \wedge \widehat{\Gamma}^{i}_{h} = -\widehat{\mathcal{R}}^{i}{}_{j}. \tag{24}$$

The h- and v-components of the torsion 2-form $\widehat{\mathcal{T}}^{\alpha} = (\widehat{\mathcal{T}}^{i}, {}^{v}\widehat{\mathcal{T}}^{i}) = \widehat{\mathbf{T}}^{\alpha}{}_{\tau\beta}\, \mathbf{e}^{\tau} \wedge \mathbf{e}^{\beta}$ from (23) is computed with components

$$\widehat{\mathcal{T}}^{i} = \widehat{C}^{i}_{jk}e^j \wedge \mathbf{e}^k, \qquad {}^{v}\widehat{\mathcal{T}}^{i} = \frac{1}{2}\, {}^{L}\Omega^{i}_{kj}e^k \wedge e^j + \left(\frac{\partial\ {}^{L}N^{i}_{k}}{\partial y^j} - \widehat{L}^{i}{}_{kj}\right)e^k \wedge \mathbf{e}^j, \tag{25}$$

where ${}^{L}\Omega^{i}_{kj}$ are coefficients of the curvature of the canonical N-connection $\check{N}^{i}_{k}$ defined by formulas similar to (16). The formulas (25) parametrize the h- and v-components of torsion $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$ in the form

$$\widehat{T}^{i}_{jk} = 0, \qquad \widehat{T}^{i}_{jc} = \widehat{C}^{i}{}_{jc}, \qquad \widehat{T}^{a}_{ij} = {}^{L}\Omega^{a}_{ij}, \qquad \widehat{T}^{a}_{ib} = e_b\left({}^{L}N^{a}_{i}\right) - \widehat{L}^{a}{}_{bi}, \qquad \widehat{T}^{a}_{bc} = 0. \tag{26}$$

It should be noted that $\widehat{\mathbf{T}}$ vanishes on h- and v-subspaces, i.e. $\widehat{T}^{i}_{jk} = 0$ and $\widehat{T}^{a}_{bc} = 0$, but certain nontrivial h-v-components induced by the nonholonomic structure are defined canonically by $\mathbf{g} = {}^{L}\mathbf{g}$ (5) and L. For convenience, in Appendix, we outline some important component formulas for the canonical d-connection which on spaces of even dimensions transform into those for the normal connection.

We compute also the curvature 2-form from (24),

$$\widehat{\mathcal{R}}^{\tau}{}_{\gamma} = \widehat{\mathbf{R}}^{\tau}{}_{\gamma\alpha\beta}\, \mathbf{e}^{\alpha} \wedge \mathbf{e}^{\beta} = \frac{1}{2}\widehat{R}^{i}{}_{jkh} e^{k} \wedge e^{h} + \widehat{P}^{i}{}_{jka} e^{k} \wedge \mathbf{e}^{a} + \frac{1}{2}\widehat{S}^{i}{}_{jcd} \mathbf{e}^{c} \wedge \mathbf{e}^{d},$$

where the nontrivial N-adapted coefficients of curvature $\widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\tau}$ of $\widehat{\mathbf{D}}$ are

$$\begin{aligned}
\widehat{R}^{i}{}_{hjk} &= \mathbf{e}_{k}\widehat{L}^{i}{}_{hj} - \mathbf{e}_{j}\widehat{L}^{i}{}_{hk} + \widehat{L}^{m}{}_{hj}\widehat{L}^{i}{}_{mk} - \widehat{L}^{m}{}_{hk}\widehat{L}^{i}{}_{mj} - \widehat{C}^{i}{}_{ha}\, {}^{L}\Omega^{a}{}_{kj},\\
\widehat{P}^{i}{}_{jka} &= e_{a}\widehat{L}^{i}{}_{jk} - \widehat{\mathbf{D}}_{k}\widehat{C}^{i}{}_{ja},\\
\widehat{S}^{a}{}_{bcd} &= e_{d}\widehat{C}^{a}{}_{bc} - e_{c}\widehat{C}^{a}{}_{bd} + \widehat{C}^{e}{}_{bc}\widehat{C}^{a}{}_{ed} - \widehat{C}^{e}{}_{bd}\widehat{C}^{a}{}_{ec}.
\end{aligned} \tag{27}$$

Contracting the first and forth indices $\widehat{\mathbf{R}}_{\beta\gamma} = \widehat{\mathbf{R}}^{\alpha}{}_{\beta\gamma\alpha}$, we get the N-adapted coefficients for the Ricci tensor $\widehat{\mathbf{R}}_{\beta\gamma} = (\widehat{R}_{ij}, \widehat{R}_{ia}, \widehat{R}_{ai}, \widehat{R}_{ab})$. The scalar curvature ${}^{L}R = \widehat{R}$ of $\widehat{\mathbf{D}}$ is

$${}^{L}R = {}^{L}\mathbf{g}^{\beta\gamma}\widehat{\mathbf{R}}_{\beta\gamma} = \mathbf{g}^{\beta'\gamma'}\widehat{\mathbf{R}}_{\beta'\gamma'}. \tag{28}$$

The normal d-connection $\widehat{\mathbf{D}}$ (21) defines a canonical almost symplectic d-connection, $\widehat{\mathbf{D}} \equiv {}_{\theta}\widehat{\mathbf{D}}$, which is N-adapted to the effective Lagrange and, related, almost symplectic structures, i.e. it preserves under parallelism the splitting (8), ${}_{\theta}\widehat{\mathbf{D}}_{\mathbf{X}}\ {}^{L}\theta = {}_{\theta}\widehat{\mathbf{D}}_{\mathbf{X}}\ \theta = 0$ and its torsion is constrained to satisfy the conditions $\widehat{T}^{i}_{jk} = \widehat{T}^{a}_{bc} = 0$.

In the canonical approach to the general relativity theory, one works with the Levi Civita connection $\bigtriangledown = \{{}_{\shortmid}\Gamma^{\alpha}_{\beta\gamma}\}$ which is uniquely derived following the conditions ${}_{\shortmid}\mathcal{T} = 0$ and $\bigtriangledown\mathbf{g} = 0$. This is a linear connection but not a d-connection because $\bigtriangledown$ does not preserve (8) under parallelism. Both linear connections $\bigtriangledown$ and $\widehat{\mathbf{D}} \equiv {}_{\theta}\widehat{\mathbf{D}}$ are uniquely defined in metric compatible forms by the same metric structure $\mathbf{g}$ (1). The second one contains nontrivial d-torsion components $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}$ (26), induced effectively by an equivalent Lagrange metric $\mathbf{g} = {}^{L}\mathbf{g}$ (5) and adapted both to the N-connection ${}^{L}\mathbf{N}$, see (3) and (8), and almost symplectic ${}^{L}\theta$ (19) structures L.

Any geometric construction for the normal d-connection $\widehat{\mathbf{D}}(\theta)$ can be re-defined by the Levi Civita connection, and inversely, using the formula

$${}_{\shortmid}\Gamma^{\gamma}{}_{\alpha\beta}(\theta) = \widehat{\Gamma}^{\gamma}{}_{\alpha\beta}(\theta) + {}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}(\theta), \tag{29}$$

where the both connections ${}_{\shortmid}\Gamma^{\gamma}{}_{\alpha\beta}(\theta)$ and $\widehat{\Gamma}^{\gamma}{}_{\alpha\beta}(\theta)$ and the distorsion tensor ${}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}(\mathbf{g})$ with N-adapted coefficients (for the normal d-connection ${}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}(\mathbf{g})$ is proportional to $\widehat{\mathbf{T}}^{\alpha}_{\beta\gamma}(\mathbf{g})$ (26)), see formulas (A.3). In this work, we emphasize if it is necessary the functional dependence of certain geometric objects on a d-metric $(\mathbf{g})$, or its canonical almost symplectic equivalent (θ) for tensors and connections completely defined by the metric structure.[7]

2.4 An Almost Symplectic Formulation of General Relativity

Having chosen a canonical almost symplectic d-connection, we compute the Ricci d-tensor $\widehat{\mathbf{R}}_{\beta\gamma}$ and the scalar curvature ${}^{L}R$, see formulas (28)). Then, we can postulate in a straight-

[7] See Appendix on similar deformation properties of fundamental geometric objects.

forward form the field equations

$$\widehat{\mathbf{R}}^{\underline{\alpha}}_{\ \beta} - \frac{1}{2}({}^{L}R + \lambda)\mathbf{e}^{\underline{\alpha}}_{\ \beta} = 8\pi G \mathbf{T}^{\underline{\alpha}}_{\ \beta}, \tag{30}$$

where $\widehat{\mathbf{R}}^{\underline{\alpha}}{}_{\beta} = \mathbf{e}^{\underline{\alpha}}_{\ \gamma}\ \widehat{\mathbf{R}}^{\ \gamma}{}_{\beta}$, $\mathbf{T}^{\underline{\alpha}}_{\ \beta}$ is the effective energy-momentum tensor, λ is the cosmological constant, G is the Newton constant in the units when the light velocity $c = 1$, and the coefficients $\mathbf{e}^{\underline{\alpha}}_{\ \beta}$ of vierbein decomposition $\mathbf{e}_{\ \beta} = \mathbf{e}^{\underline{\alpha}}_{\ \beta}\partial/\partial u^{\underline{\alpha}}$ are defined by the N-coefficients of the N-elongated operator of partial derivation, see (10). But the equations (30) for the canonical $\widehat{\Gamma}^{\gamma}{}_{\alpha\beta}(\theta)$ are not equivalent to the Einstein equations in general relativity written for the Levi–Civita connection ${}_{\shortmid}\Gamma^{\gamma}{}_{\alpha\beta}(\theta)$ if the tensor $\mathbf{T}^{\underline{\alpha}}_{\ \beta}$ does not include contributions of ${}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}(\theta)$ in a necessary form.

Introducing the absolute antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$ and the effective source 3-form

$$\mathcal{T}_{\beta} = \mathbf{T}^{\underline{\alpha}}_{\ \beta}\ \epsilon_{\underline{\alpha\beta\gamma\delta}} du^{\underline{\beta}} \wedge du^{\underline{\gamma}} \wedge du^{\underline{\delta}}$$

and expressing the curvature tensor $\widehat{\mathcal{R}}^{\tau}{}_{\gamma} = \widehat{\mathbf{R}}^{\tau}{}_{\gamma\alpha\beta}\ \mathbf{e}^{\alpha} \wedge\ \mathbf{e}^{\beta}$ of $\widehat{\Gamma}^{\alpha}{}_{\beta\gamma} = {}_{\shortmid}\Gamma^{\alpha}{}_{\beta\gamma} - {}_{\shortmid}\widehat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$ as $\widehat{\mathcal{R}}^{\tau}{}_{\gamma} = {}_{\shortmid}\mathcal{R}^{\tau}{}_{\gamma} - {}_{\shortmid}\widehat{\mathcal{Z}}^{\tau}{}_{\gamma}$, where ${}_{\shortmid}\mathcal{R}^{\tau}{}_{\gamma} = {}_{\shortmid}R^{\tau}{}_{\gamma\alpha\beta}\ \mathbf{e}^{\alpha} \wedge \mathbf{e}^{\beta}$ is the curvature 2-form of the Levi–Civita connection ∇ and the distorsion of curvature 2-form $\widehat{\mathcal{Z}}^{\tau}{}_{\gamma}$ is defined by $\widehat{\mathbf{Z}}^{\alpha}{}_{\beta\gamma}$, see (29), we derive (30) (varying the action on components of $\mathbf{e}_{\ \beta}$, see details in [14]). The gravitational field equations are represented as 3-form equations,

$$\epsilon_{\alpha\beta\gamma\tau}\left(\mathbf{e}^{\alpha} \wedge \widehat{\mathcal{R}}^{\beta\gamma} + \lambda\mathbf{e}^{\alpha} \wedge\ \mathbf{e}^{\beta} \wedge\ \mathbf{e}^{\gamma}\right) = 8\pi G\mathcal{T}_{\tau}, \tag{31}$$

when $\mathcal{T}_{\tau} = {}^{m}\mathcal{T}_{\tau} + {}^{Z}\widehat{\mathcal{T}}_{\tau}$,

$$\begin{aligned} {}^{m}\mathcal{T}_{\tau} &= {}^{m}\mathbf{T}^{\underline{\alpha}}_{\ \tau}\epsilon_{\underline{\alpha\beta\gamma\delta}}du^{\underline{\beta}} \wedge du^{\underline{\gamma}} \wedge du^{\underline{\delta}}, \\ {}^{Z}\mathcal{T}_{\tau} &= (8\pi G)^{-1}\ \widehat{\mathcal{Z}}^{\underline{\alpha}}_{\ \tau}\epsilon_{\underline{\alpha\beta\gamma\delta}}du^{\underline{\beta}} \wedge du^{\underline{\gamma}} \wedge du^{\underline{\delta}}, \end{aligned}$$

where ${}^{m}\mathbf{T}^{\underline{\alpha}}_{\ \tau}$ is the matter tensor field. The above mentioned equations are equivalent to the usual Einstein equations for the Levi–Civita connection ∇,

$${}_{\shortmid}\mathbf{R}^{\underline{\alpha}}_{\ \beta} - \frac{1}{2}({}_{\shortmid}R + \lambda)\mathbf{e}^{\underline{\alpha}}_{\ \beta} = 8\pi G\ {}^{m}\mathbf{T}^{\underline{\alpha}}_{\ \beta}.$$

If former geometric constructions in general relativity were related to frame and coordinate form invariant transforms, various purposes in geometric modelling of physical interactions and quantization request application of more general classes of transforms. For such generalizations, the linear connection structure is deformed (in a unique/canonical form following well defined geometric and physical principles) and there are considered nonholonomic spacetime distributions. All geometric and physical information for any data 1) $[\mathbf{g}, {}_{\shortmid}\Gamma^{\gamma}{}_{\alpha\beta}(\mathbf{g})]$ are transformed equivalently for canonical constructions with 2) $[\mathbf{g} = {}^{L}\mathbf{g}, \mathbf{N}, \widehat{\Gamma}^{\gamma}{}_{\alpha\beta}(\mathbf{g})]$, which allows us to provide an effective Lagrange interpretation of the Einstein gravity, or 3) $[\theta = {}^{L}\theta,\ {}_{\theta}\widehat{\Gamma}^{\gamma}{}_{\alpha\beta} = \widehat{\Gamma}^{\gamma}{}_{\alpha\beta}, \mathbf{J}(\theta)]$, for an almost Kähler model of general relativity. The canonical almost symplectic form θ (20) represents the "original" metric $\mathbf{g}$ (1) equivalently in a "nonsymmetric" form. Any deformations of such structures, in the framework of general relativity or quantized models and generalizations, result in more general classes of nonsymmetric metrics.

3 Nonsymmetric Gravity Theories with Nonholonomic Distributions

In this section, we follow the geometric conventions and results from [1]. We outline some basic definitions and formulas from the geometry of nonholonomic manifolds enabled with nonlinear connection and general nonsymmetric structure and introduce a general Lagrangian for nonsymmetric gravity theories and corresponding nonholonomic distributions.

3.1 On Geometry of N-Anholonomic Manifolds

In this paper, we also consider gravity models on spaces $(\check{g}_{ij}, \mathbf{V}^{n+n}, \mathbf{N})$ when the h-subspace is enabled with a nonsymmetric tensor field (metric) $\check{g}_{ij} = g_{ij} + a_{ij}$, where the symmetric part $g_{ij} = g_{ji}$ is nondegenerated and $a_{ij} = -a_{ji}$. A d-metric $\check{g}_{ij}(x,y)$ is of index k if there are satisfied the properties: 1. $\det|g_{ij}| \neq 0$ and 2. $\mathrm{rank}|a_{ij}| = n - k = 2p$, for $0 \leq k \leq n$. By g^{ij} we note the reciprocal (inverse) to g_{ij} d-tensor field. The matrix a_{ij} is not invertible unless for $k = 0$.

For $k > 0$ and a positive definite $g_{ij}(x,y)$, on each domain of local chart there exists k d-vector fields $\xi^i_{i'}$, where $i = 1, 2, \ldots, n$ and $i' = 1, \ldots, k$ with the properties $a_{ij}\xi^j_{j'} = 0$ and $g_{ij}\xi^i_{i'}\xi^j_{j'} = \delta_{i'j'}$. If g_{ij} is not positive definite, we shall assume the existence of k linearly independent d-vector fields with such properties.

The metric properties on $\mathbf{V}^{n+n}$ are supposed to be defined by d-tensor

$$\check{\mathbf{g}} = \mathbf{g} + \mathbf{a} = \check{\mathbf{g}}_{\alpha\beta}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = \check{g}_{ij}e^i \otimes e^j + \check{g}_{ab}\mathbf{e}^a \otimes \mathbf{e}^b, \tag{32}$$

$$\mathbf{g} = \mathbf{g}_{\alpha\beta}\mathbf{e}^\alpha \otimes \mathbf{e}^\beta = g_{ij}e^i \otimes e^j + g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b, \tag{33}$$

$$\mathbf{a} = a_{ij}e^i \wedge e^j + a_{cb}\mathbf{e}^c \wedge \mathbf{e}^b,$$

where the v-components $\check{g}_{ab}$ are defined by the same coefficients as $\check{g}_{ij}$. With respect to a coordinate local cobasis $du^\alpha = (dx^i, dy^a)$, we have equivalently $\mathbf{g} = \underline{g}_{\alpha\beta}du^\alpha \otimes du^\beta$, where

$$\underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N^a_i N^b_j g_{ab} & N^e_j g_{ae} \\ N^e_i g_{be} & g_{ab} \end{bmatrix}. \tag{34}$$

A h-v-metric on a N-anholonomic manifold is a second rank d-tensor of type (32). One considers matrices (see details in [1]) $\widehat{g} = (g_{ij})$, $\widehat{\xi} = (\xi^i_{i'})$, $\widehat{l} = (l^i_j)$, $\widehat{\eta} = (\eta^{i'}_i)$, $\widehat{m} = (m^i_j)$, $\widehat{\delta}' = (\delta_{i'j'})$, $\widehat{\delta} = (\delta^i_j)$, where the skew symmetric matrices $\widehat{a} = (a_{ij})$ and $\check{a} = (\check{a}^{ij})$ do not depend on the choice of $\widehat{\xi}$ and uniquely defined by $\widehat{a}\check{a} = {}^t\widehat{m}$ and $\widehat{l}\check{a} = 0$ on $\mathbf{V}^{n+n}$.

In general, the concept of linear connection (adapted or not adapted to a N-connection structure) is independent from the concept of metric (symmetric or nonsymmetric). A distinguished connection (d-connection) $\mathbf{D}$ on $\mathbf{V}$ is a N-adapted linear connection, preserving by parallelism the vertical and horizontal distribution (8). In local form, $\mathbf{D} = ({}^hD, {}^vD)$ is given by its coefficients $\Gamma^\gamma_{\ \alpha\beta} = (L^i_{jk}, L^a_{bk}, C^i_{jc}, C^a_{bc})$, where ${}^hD = (L^i_{jk}, L^a_{bk})$ and ${}^vD = (C^i_{jc}, C^a_{bc})$ are respectively the covariant h- and v-derivatives. For any d-connection, we can compute the torsion, curvature and Ricci tensors and scalar curvature, see Appendix.

A normal d-connection ${}_n\mathbf{D}$ is compatible with the almost complex structure $\mathbf{J}$ (14), i.e. satisfies the condition

$${}_n\mathbf{D}_\mathbf{X}\mathbf{J} = 0, \tag{35}$$

for any d-vector $\mathbf{X}$ on $\mathbf{V}^{n+n}$. The operator ${}_n\mathbf{D}$ is characterized by a pair of local coefficients ${}_n\Gamma^\gamma_{\ \alpha\beta} = ({}_nL^i_{jk},\ {}_nC^a_{bc})$ defined by conditions ${}_n\mathbf{D}_{\mathbf{e}_k}(\mathbf{e}_j) = {}_nL^i_{jk}\mathbf{e}_i$, ${}_n\mathbf{D}_{\mathbf{e}_k}(e_a) =$

${}_nL^b_{ak}e_b$, ${}_n\mathbf{D}_{e_c}(\mathbf{e}_j) = {}_nC^i_{jc}\mathbf{e}_i$, ${}_n\mathbf{D}_{e_c}(e_a) = {}_nC^b_{ac}e_b$, where ${}_nL^i_{jk} = {}_nL^b_{ak}$ for $j = a, i = b$, and ${}_nC^i_{jc} = {}_nC^b_{ac}$, $j = a, i = b$. Here we emphasize that the normal d-connection ${}_n\mathbf{D}$ is different from $\widehat{\mathbf{D}}$ (21) (the first one is defined for a space with nonsymmetric metrics, but for the second one the metrics must by symmetric).

A d-connection $\mathbf{D} = \{\Gamma^\gamma_{\ \alpha\beta}\}$ is compatible with a nonsymmetric d-metric $\check{\mathbf{g}}$ if

$$\mathbf{D}_k\check{g}_{ij} = 0 \quad \text{and} \quad \mathbf{D}_a\check{g}_{ij} = 0. \tag{36}$$

For a d-metric (32), the equations (36) are

$$\mathbf{D}_k g_{ij} = 0, \qquad \mathbf{D}_a g_{bc} = 0, \qquad \mathbf{D}_k a_{ij} = 0, \qquad \mathbf{D}_e a_{bc} = 0. \tag{37}$$

The set of d-connections $\{\mathbf{D}\}$ satisfying the conditions $\mathbf{D_X g} = \mathbf{0}$ for a given $\mathbf{g}$ is defined by formulas

$$L^i_{\ jk} = \widehat{L}^i_{jk} + {}^-O^{ei}_{km}\mathbf{X}^m_{ej}, \qquad L^a_{\ bk} = \widehat{L}^a_{bk} + {}^-O^{ca}_{bd}\mathbf{Y}^d_{ck},$$
$$C^i_{\ jc} = \widehat{C}^i_{jc} + {}^+O^{mi}_{jk}\mathbf{X}^k_{mc}, \qquad C^a_{\ bc} = \widehat{C}^a_{bc} + {}^+O^{ea}_{bd}\mathbf{Y}^d_{ec},$$

where

$${}^{\pm}O^{ih}_{jk} = \frac{1}{2}(\delta^i_j\delta^h_k \pm g_{jk}g^{ih}), \qquad {}^{\pm}O^{ca}_{bd} = \frac{1}{2}(\delta^c_b\delta^a_d \pm g_{bd}g^{ca}) \tag{38}$$

are the so-called the Obata operators; $\mathbf{X}^m_{ej}$, $\mathbf{X}^k_{mc}$, $\mathbf{Y}^d_{ck}$ and $\mathbf{Y}^d_{ec}$ are arbitrary d-tensor fields and $\widehat{\Gamma}^\gamma_{\ \alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$, with

$$\begin{aligned}
\widehat{L}^i_{jk} &= \frac{1}{2}g^{ir}\left(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}\right),\\
\widehat{L}^a_{bk} &= e_b(N^a_k) + \frac{1}{2}g^{ac}\left(e_k g_{bc} - g_{dc}\, e_b N^d_k - g_{db}\, e_c N^d_k\right),\\
\widehat{C}^i_{jc} &= \frac{1}{2}g^{ik}e_c g_{jk},\ \widehat{C}^a_{bc} = \frac{1}{2}g^{ad}\left(e_c g_{bd} + e_c g_{cd} - e_d g_{bc}\right)
\end{aligned} \tag{39}$$

is the canonical d-connections uniquely defined by the coefficients of d-metric $\mathbf{g} = [g_{ij}, g_{ab}]$ and N-connection $\mathbf{N} = \{N^a_i\}$ in order to satisfy the conditions $\widehat{\mathbf{D}}_\mathbf{X}\mathbf{g} = \mathbf{0}$ and $\widehat{T}^i_{\ jk} = 0$ and $\widehat{T}^a_{bc} = 0$ but $\widehat{T}^i_{\ ja}, \widehat{T}^a_{ji}$ and $\widehat{T}^a_{bi}$ are not zero (on definition of torsion, see Appendix; we can compute the torsion coefficients $\widehat{\mathbf{T}}^\gamma_{\ \alpha\beta}$ by introducing d-connection coefficients (39) into (A.1)).

By direct computations, we can check that for any given d-connection ${}_\circ\Gamma^\alpha_{\ \beta\gamma} = ({}_\circ L^i_{\ jk},\ {}_\circ C^a_{bc})$ and nonsymmetric d-metric $\check{\mathbf{g}} = \mathbf{g} + \mathbf{a}$ on $\mathbf{V}$ the d-connection ${}_*\Gamma^\alpha_{\ \beta\gamma} = ({}_*L^i_{\ jk},\ {}_*C^a_{bc})$, where

$${}_*L^i_{\ jk} = {}_\circ L^i_{\ jk} + \frac{1}{2}[g^{ir}\ {}_\circ D_k g_{rj} + {}^{\pm}O^{ir}_{sj}(\check{a}^{st}\ {}_\circ D_k a_{tr} + 3l^s_t\ {}_\circ D_k l^t_r - {}_\circ D_k l^s_r)]$$

$${}_*C^a_{bc} = {}_\circ C^a_{bc} + \frac{1}{2}[g^{ah}\ {}_\circ D_c g_{hb} + {}^{\pm}O^{ah}_{eb}(\check{a}^{ed}\ {}_\circ D_c a_{dh} + 3l^e_d\ {}_\circ D_c l^d_h - {}_\circ D_c l^e_h)]$$

is d-metric compatible, i.e there are satisfied the conditions ${}_*\mathbf{D}\check{\mathbf{g}} = \mathbf{0}$.

The set of d-connections $\mathbf{D} = {}_\circ\mathbf{D} + \mathbf{B}$ being generated by deformations of an arbitrary fixed d-connection ${}_\circ\mathbf{D}$ in order to be compatible with a given nonsymmetric d-metric $\check{\mathbf{g}} =$

$\mathbf{g}+\mathbf{a}$ on $\mathbf{V}$ is defined by distorsion d-tensors $\mathbf{B}=({}_h\mathbf{B},\ {}_v\mathbf{B})$ which can be computed in explicit form, see [1]. In this paper, for simplicity, we shall work with a general d-connection $\mathbf{D}$ which is compatible to $\check{\mathbf{g}}$, i.e. satisfies the conditions (37), or (36), and can be generated by a distorsion tensor $\mathbf{B}$ from $\widehat{\mathbf{D}}$ (39), or from ${}_n\mathbf{D}$ (35). We note that for certain canonical constructions the d-objects $\mathbf{D}$, ${}_\circ\mathbf{D}$, $\widehat{\mathbf{D}}$, ${}_n\mathbf{D}$ and $\mathbf{B}$ are completely defined by the coefficients of a d-metric $\check{\mathbf{g}}=\mathbf{g}+\mathbf{a}$ and $\mathbf{N}$ on $\mathbf{V}$.

Finally, it should be emphasized that because ${}_\circ\Gamma^{\alpha}{}_{\beta\gamma}=({}_\circ L^{i}_{jk},\ {}_\circ C^{a}_{bc})$ is an arbitrary d-connection, it can be chosen to be an important one for certain physical or geometrical problems. In this work, we shall consider certain exact solutions in gravity with nonholonomic variables defining a corresponding ${}_\circ\Gamma^{\alpha}{}_{\beta\gamma}$ and then deformed to nonsymmetric configurations.

3.2 General Nonsymmetric Gravity Models with d-Connections

The goal of this section is to analyze N-adapted nonholonomic nonsymmetric gravity models completely defined by a N-connection $\mathbf{N}=\{N^a_i\}$, d-metric $\check{\mathbf{g}}=\mathbf{g}+\mathbf{a}$ (32) and a metric compatible d-connection $\Gamma^{\lambda}_{\mu\nu}$.

We follow a N-adapted variational calculus, when instead of partial derivatives there are used the "N-elongated" partial derivatives $\mathbf{e}_\rho$ (10), varying independently the d-fields $\check{\mathbf{g}}=\mathbf{g}+\mathbf{a}$ and $\Gamma^{\alpha}{}_{\beta\gamma}$. In this case, $\check{a}=(\check{a}^{ij})$ does not depend on the choice of fields $\widehat{\xi}$ and we can write $\check{\mathbf{g}}^{[\rho\sigma]}=\check{\mathbf{a}}^{\rho\sigma}=[\check{a}^{ij},\check{a}^{cb}]$, where $\check{a}^{ij}=-\check{a}^{ji}$ and $\check{a}^{cb}=-\check{a}^{bc}$. We shall work with d-connections, $\mathbf{W}^{\lambda}_{\mu\nu}\doteqdot\Gamma^{\lambda}_{\mu\nu}-\frac{2}{3}\delta^{\lambda}_{\mu}\,\mathbf{W}_\nu$, where $\mathbf{W}_\nu=\frac{1}{2}(\mathbf{W}^{\lambda}_{\mu\lambda}-\mathbf{W}^{\lambda}_{\lambda\mu})$, which means that $\Gamma^{\alpha}{}_{[\beta\gamma]}=0$. This defines a covariant derivative of type ${}_W\mathbf{D}_\gamma\check{\mathbf{g}}_{\alpha\beta}=\mathbf{e}_\gamma\check{\mathbf{g}}_{\alpha\beta}-\mathbf{W}^{\tau}{}_{\alpha\gamma}\check{\mathbf{g}}_{\tau\beta}-\mathbf{W}^{\tau}{}_{\beta\gamma}\check{\mathbf{g}}_{\alpha\tau}$. We also can compute $\mathbf{P}_{\mu\nu}\doteqdot{}_W\mathbf{R}^{\lambda}{}_{\lambda\mu\nu}=\mathbf{e}_\mu\mathbf{W}^{\lambda}{}_{\lambda\nu}-\mathbf{e}_\nu\mathbf{W}^{\lambda}{}_{\lambda\mu}$, where ${}_W\mathbf{R}^{\lambda}{}_{\lambda\mu\nu}$ is computed following formulas (A.2) with d-connection $\mathbf{W}$ instead of Γ. The corresponding to $\mathbf{W}$ and Γ Ricci d-tensors, are related by formulas

$$
{}_W\mathbf{R}_{\mu\nu}={}_\Gamma\mathbf{R}_{\mu\nu}+\frac{2}{3}\mathbf{e}_{[\nu}\,\mathbf{W}_{\mu]},
$$

where ${}_\Gamma\mathbf{R}_{\mu\nu}$ is computed for the symmetric part of metric. The variables of this generalized theory, with gravitational constant $(16\pi G_N)^{-1}=1$, are parametrized:

$$
\begin{aligned}
\check{\mathbf{g}}_{\mu\nu}&=\mathbf{g}_{\mu\nu}+\mathbf{a}_{\mu\nu}+\dots,\quad \text{full, nonsymmetric d-metric;}\\
\check{\mathbf{g}}_{(\mu\nu)}&=\frac{1}{2}(\check{\mathbf{g}}_{\mu\nu}+\check{\mathbf{g}}_{\nu\mu})\approx\mathbf{g}_{\mu\nu},\quad \text{symmetric d-metric;}\\
\check{\mathbf{g}}_{[\mu\nu]}&=\frac{1}{2}(\check{\mathbf{g}}_{\mu\nu}-\check{\mathbf{g}}_{\nu\mu})\approx\mathbf{a}_{\mu\nu},\quad \text{antisymmetric d-metric;}\\
\check{\mathbf{g}}_{\mu\alpha}\check{\mathbf{g}}^{\mu\beta}&=\check{\mathbf{g}}_{\alpha\mu}\check{\mathbf{g}}^{\beta\mu}=\delta^{\beta}_{\alpha}\neq\check{\mathbf{g}}_{\alpha\mu}\check{\mathbf{g}}^{\mu\beta};\\
\mathbf{W}^{\alpha}{}_{\beta\gamma}&\doteqdot\Gamma^{\alpha}{}_{\beta\gamma}-\frac{2}{3}\delta^{\alpha}_{\beta}\mathbf{W}_{\ \gamma},\quad \text{full, nonsymmetric d-connection;}\\
\mathbf{W}_{\ \beta}&\doteqdot\mathbf{W}^{\alpha}{}_{[\beta\alpha]}.
\end{aligned}
$$

We shall use a nonholonomic generalization of the Lagrangian from [34],

$$
\begin{aligned}
\mathcal{L}=\sqrt{-\check{\mathbf{g}}}\check{\mathbf{g}}^{\mu\nu}[&{}_W\mathbf{R}_{\mu\nu}+a_1\mathbf{P}_{\mu\nu}+a_2\mathbf{e}_{[\mu}\,\mathbf{W}_{\nu]}+b_1\,{}_W\mathbf{D}_\gamma\mathbf{W}^{\gamma}{}_{[\mu\nu]}\\
&+b_2\mathbf{W}^{\lambda}{}_{[\mu\alpha]}\mathbf{W}^{\alpha}{}_{[\lambda\nu]}+b_3\mathbf{W}^{\lambda}{}_{[\mu\nu]}\mathbf{W}_\lambda+\check{\mathbf{g}}^{\lambda\delta}\check{\mathbf{g}}_{\alpha\beta}(c_1\mathbf{W}^{\alpha}{}_{[\mu\lambda]}\mathbf{W}^{\beta}{}_{[\nu\delta]}
\end{aligned}
$$

$$+ c_2 \mathbf{W}^{\alpha}{}_{[\mu\nu]} \mathbf{W}^{\beta}{}_{[\lambda\delta]} + c_3 \mathbf{W}^{\alpha}{}_{[\mu\delta]} \mathbf{W}^{\beta}{}_{[\nu\lambda]} + d_1 \mathbf{W}_{\mu} \mathbf{W}_{\nu} + 2\Lambda)], \tag{40}$$

where the parameters a_1, a_2, etc. are certain constants and Λ is the cosmological constant. One should fix certain values of such constants and take $\mathbf{W}^{\alpha}{}_{\beta\gamma}$ to be defined by a general affine (in particular, Levi–Civita) connection, in order to get different Moffat or other models of noncommutative gravity theory.

3.3 Linearization to Symmetric Anholonomic Backgrounds

We prove that for general nonsymmetric metrics defined on nonholonomic manifolds and corresponding nonholonomic deformations and linearization a class of general Lagrangians for nonsymmetric gravity theory can be transformed into stable Lagrangians similar to those used in σ-model and anholonomic and/or noncommutative corrections to general relativity. We follow the geometric formalism elaborated in [1, 39, 40] and reconsider the results of works [31, 32] for nonholonomic spaces enabled both with nonlinear connection and nonsymmetric metric structures.

Let us consider an expansion of the Lagrangian (40) for $\check{\mathbf{g}} = \mathbf{g} + \mathbf{a}$ around a background spacetime defined by a symmetric metric $\mathbf{g} = \{\mathbf{g}_{\alpha\beta}\}$ and a metric compatible d-connection ${}^{b}\Gamma^{\alpha}{}_{\beta\gamma}$ defined by $\mathbf{N}$ and $\mathbf{g}$ (it can be a normal, canonical d-connection, Cartan or another one) and denote $\check{\mathbf{g}}_{[\alpha\beta]} = \mathbf{a}_{\alpha\beta}$. We use decompositions of type

$$\begin{aligned} \check{\mathbf{g}}_{\alpha\beta} &= \mathbf{g}_{\alpha\beta} + {}^{1}\mathbf{g}_{\alpha\beta} + \dots, \qquad \mathbf{a}_{\alpha\beta} = {}^{1}\mathbf{a}_{\alpha\beta} + {}^{2}\mathbf{a}_{\alpha\beta} \dots, \\ \Gamma^{\alpha}{}_{\beta\gamma} &= {}^{b}\Gamma^{\alpha}{}_{\beta\gamma} + {}^{1}\Gamma^{\alpha}{}_{\beta\gamma} + \dots, \qquad \mathbf{W}_{\mu} = {}^{1}\mathbf{W}_{\mu} + {}^{2}\mathbf{W}_{\mu} + \dots \end{aligned} \tag{41}$$

when, re-defining ${}^{1}\mathbf{a}_{\alpha\beta} \to \mathbf{a}_{\alpha\beta}$, ${}^{2}\mathbf{a}_{\alpha\beta} \sim \mathbf{a}_{..} \cdot \mathbf{a}_{..}$, one holds

$$\begin{aligned} \check{\mathbf{g}}_{\mu\nu} &= \mathbf{g}_{\mu\nu} + \mathbf{a}_{\mu\nu} + \rho \mathbf{a}_{\mu\alpha} \mathbf{a}^{\alpha}{}_{\nu} + \sigma \mathbf{a}^2 \mathbf{g}_{\mu\nu} + O(\mathbf{a}^3), \\ \check{\mathbf{g}}^{\mu\nu} &= \mathbf{g}^{\mu\nu} + \mathbf{a}^{\mu\nu} + (1-\rho) \mathbf{a}^{\mu\alpha} \mathbf{a}_{\alpha}{}^{\nu} + \sigma \mathbf{a}^2 \mathbf{g}_{\mu\nu} + O(\mathbf{a}^3), \end{aligned} \tag{42}$$

which implies that $\sqrt{|\check{\mathbf{g}}_{\mu\nu}|} = \sqrt{|\mathbf{g}_{\mu\nu}|}[1 + \frac{1}{2}(\frac{1}{2} - \rho + 4\sigma)\mathbf{a}^2]$, for $\mathbf{a}^2 = \mathbf{a}_{\mu\alpha}\mathbf{a}^{\mu\alpha}$, where $\mathbf{g}_{\mu\nu}$ and its inverse $\mathbf{g}^{\mu\nu}$ are used to raise and lower indices. Following a N-adapted calculus with "N-elongated" partial differential and differential operators (see (10) and (11)) instead of usual partial derivatives and local coordinate (co) bases, similarly to constructions in Appendix to [1], we get from (40) (up to the second order approximations on $\mathbf{a}$) the effective Lagrangian

$$\begin{aligned} \mathcal{L} = \sqrt{-g}\Big[{}^{s}R + 2\Lambda - \frac{1}{12}\mathbf{H}^2 + \left(\frac{1}{4}\mu^2 + \beta\, {}^{s}R\right)\mathbf{a}^2 \\ - \alpha \mathbf{R}_{\mu\nu} \mathbf{a}^{\mu\alpha} \mathbf{a}_{\alpha}{}^{\nu} - \gamma \mathbf{R}_{\mu\alpha\nu\beta} \mathbf{a}^{\mu\nu} \mathbf{a}^{\alpha\beta} \Big] + O(\mathbf{a}^3), \end{aligned} \tag{43}$$

where the effective gauge field (absolutely symmetric torsion) is

$$\mathbf{H}_{\alpha\beta\gamma} = \mathbf{e}_{\alpha} \mathbf{a}_{\beta\gamma} + \mathbf{e}_{\beta} \mathbf{a}_{\gamma\alpha} + \mathbf{e}_{\gamma} \mathbf{a}_{\alpha\beta}, \tag{44}$$

with an effective mass for $\mathbf{a}_{\beta\gamma}$, $\mu^2 = 2\Lambda(1 - 2\rho + 8\sigma)$, when the curvature d-tensor $\mathbf{R}_{\mu\alpha\nu\beta}$, Ricci d-tensor $\mathbf{R}_{\mu\nu}$ and scalar curvature ${}^{s}R$ are correspondingly computed following formulas (A.2) and (28), and the constants from (40) and (42) are re-defined following formulas (A.4) in Appendix.

If in the effective Lagrangian (43) we take instead of a metric compatible d-connection $\Gamma^{\alpha}_{\ \beta\gamma}$ the Levi–Civita connection $_{\shortmid}\Gamma^{\alpha}_{\ \beta\gamma}$, we get the formula (A29) in [31] for nonsymmetric gravitational interactions modelled on a (pseudo) Riemannian background. It exists a theorem proven by van Nieuwenhuizen [46] stating that in flat space the only consistent action for a massive antisymmetric tensor field is of the form

$$
^{fl}\mathcal{L} = -\frac{1}{12}\mathbf{H}^2 + \frac{1}{4}\mu^2\mathbf{a}^2 + O(\mathbf{a}^3), \tag{45}
$$

for $\mathbf{a}^2 = \mathbf{a}^{\mu\nu}\mathbf{a}_{\mu\nu}$. A rigorous study provided in [31] proves that $\gamma = 0$, see (43), is not allowed in nonsymmetric gravity theories extended nearly a Schwarzschild background because in such a case it is not possible to solve in a compatible form the conditions (A.5) for $\gamma = \Xi = 0$.

A quite general solution of the problem of instability in nonsymmetric gravity theories found by Janssen and Prokopec is to compensate the term with $\gamma \neq 0$ in (43). To do this, we can constrain such a way the nonholonomic frame dynamics[8] when we get for decompositions of a noncommutative gravity theory with respect to any general relativity background an effective Lagrangian without coupling of spacetime curvature tensors with nonsymmetric tensor $\mathbf{b}^{\mu\alpha}$ (i.e. without a term of type $\gamma\, _{\shortmid}R_{\mu\alpha\nu\beta}\mathbf{b}^{\mu\nu}\mathbf{b}^{\alpha\beta}$),

$$
^{E}\mathcal{L} = \sqrt{-g}\left[_{\shortmid}R + 2\, _{\shortmid}\Lambda - \frac{1}{12}\,\mathbf{H}^2 + \left(\frac{1}{4}\mu^2 + \beta\, _{\shortmid}R\right)\mathbf{b}^2 - \alpha\, _{\shortmid}R_{\mu\nu}\mathbf{b}^{\mu\alpha}\mathbf{b}_{\alpha}^{\ \nu}\right] + O(b^3). \tag{46}
$$

In this formula $_{\shortmid}R$ and $_{\shortmid}R_{\mu\nu}$ are respectively the scalar curvature and the Ricci tensor computed for $_{\shortmid}\Gamma^{\alpha}_{\ \beta\gamma}$ and $_{\shortmid}\Lambda$ is an effective cosmological constant with possible small polarizations depending on u^{α}.

We show how for a class of nonholonomic deformations of general relativity backgrounds, we get effective Lagrangians which seem to have a good flat spacetime limit of type (45):

Let us consider $N_i^a \approx \mathring{\varepsilon}^2\, n_i^a$ and $\mathbf{a}^{\mu\alpha} \approx \mathring{\varepsilon}\,\mathbf{b}^{\mu\alpha}$ and take $^{b}\Gamma^{\alpha}_{\ \beta\gamma} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$ (39) in decomposition for d-connection (41), where $\mathring{\varepsilon}$ is a small parameter, which results (following formulas (29), (44) and (10)) in deformations of type

$$
\begin{aligned}
_{\shortmid}\Gamma^{\gamma}_{\ \alpha\beta} &= \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} + \mathring{\varepsilon}^2\, _{\shortmid}\mathring{z}^{\gamma}_{\ \alpha\beta}(n_i^a)\ldots, \\
^{s}R &= {_{\shortmid}R} + \mathring{\varepsilon}^2\, _{\shortmid}\mathring{z}(n_i^a)\ldots, \mathbf{H}^2 = \mathring{\varepsilon}^2\, _{\shortmid}H(\mathbf{b}^{\mu\alpha}),
\end{aligned} \tag{47}
$$

where $_{\shortmid}H(\mathbf{b}^{\mu\alpha})$ is computed by formula (44) with $\mathbf{e}_{\alpha} \to \partial_{\alpha}$ and $\mathbf{a}_{\beta\gamma} \to \mathbf{b}_{\beta\gamma}$ and the functionals $_{\shortmid}z^{\gamma}_{\ \alpha\beta}(g_{ij,}g_{ab}, n_i^a)$ and $_{\shortmid}\mathring{z}(g_{ij,}g_{ab}, n_i^a)$ can be computed by introducing (A.3) into respective formulas for connections and scalar curvature. Introducing values (47) into (45) and identifying $_{\shortmid}\Lambda \approx \Lambda$, we get that $\mathcal{L} \to\ ^{E}\mathcal{L}$ if and only if

$$
_{\shortmid}z(g_{ij,}g_{ab}, \mathring{n}_i^a) = \gamma\ _{\shortmid}R_{\mu\alpha\nu\beta}\mathbf{b}^{\mu\nu}\mathbf{b}^{\alpha\beta}. \tag{48}
$$

The left part of this equation is defined by the quadratic $\mathring{\varepsilon}^2$ deformation of scalar curvature, from $_{\shortmid}R$ to ^{s}R, relating algebraically the coefficients g_{ij}, h_{ab} and n_i^a and their partial derivatives. We do not provide in this work the cumbersome formula for $_{\shortmid}\mathring{z}(g_{ij,}g_{ab}, n_i^a)$ in

[8] In explicit form, we have to impose certain constraints on coefficients N_i^a from (34) and (33), see the end of this section.

the case of general nonholonomic or Einstein gravity backgrounds, but we shall compute it explicitly and solve (48) for an ellipsoidal background in next section. Here we emphasize that in theories with zero cosmological constant we have to consider ${}_{\shortmid}\Lambda \approx \Lambda = 0$.

We conclude that we are able to generate stable nonsymmetric gravity models on backgrounds with small nonholonomic frame and nonsymmetric metric deformations if the conditions (48) are satisfied. This induces a small locally anisotropic polarization of the cosmological constant. Having stabilized the gravitational interactions with the nonsymmetric components of metric, for certain gravitational configurations with another small parameter $\varepsilon \to 0$, we get certain backgrounds in general relativity (for instance, the Schwarzschild one). For generic nonlinear theories, such as nonsymmetric gravity theories and the Einstein gravity, the procedures of constraining certain nonlinear solutions in order to get stable configurations and taking smooth limits on a small parameter resulting in holonomic backgrounds are not commutative.

Finally, we note that we can use similar decompositions of type (47) to transform an arbitrary metric compatible d-connection $\Gamma^{\gamma}_{\ \alpha\beta}$ to $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$, and/or to introduce two small parameters for deformations of type $\Gamma^{\gamma}_{\ \alpha\beta} \to \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} \to {}_{\shortmid}\Gamma^{\gamma}_{\ \alpha\beta}$. We shall use this approach in the next section.

4 Stability of Stationary Ellipsoidal Solutions

The effective gravitational field equations for nonsymmetric metrics on symmetric nonholonomic backgrounds are derived. We also analyze a class of solutions in nonsymmetric gravity theories on a nonholonomic ellipsoidal background. For vanishing eccentricity, such solutions have nontrivial limits to Schwarzschild configurations.

4.1 Field Equations with Nonholonomic Backgrounds

The field equations derived from an effective Lagrangian (43) for a d-connection $\Gamma^{\gamma}_{\ \alpha\beta}$ are

$$\left(\sqrt{|\mathbf{g}_{\mu\nu}|}\right)^{-1}\mathbf{e}_{\alpha}(\sqrt{|\mathbf{g}_{\mu\nu}|}\mathbf{H}^{\alpha\beta\nu}) + (\mu^2 + 4\beta\, {}^{s}R)\mathbf{a}^{\beta\nu}$$
$$+ 4\alpha \mathbf{a}^{\alpha(\nu}\mathbf{R}^{\beta)}_{\alpha} + 4\gamma \mathbf{a}^{\alpha\tau}\mathbf{R}^{\beta\ \ \ \nu}_{\ \alpha\tau} + \mathcal{O}(\mathbf{a}^2) = 0,$$
$$\mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\, {}^{s}R - \Lambda \mathbf{g}_{\mu\nu} + \mathcal{O}(\mathbf{a}^2) = 0.$$

We shall work with two-parameter, deformations of nonlinear and linear connections, respectively of $N^a_i \approx \varepsilon n^a_i + \mathring{\varepsilon}^2 \mathring{n}^a_i + \ldots$ and $\mathbf{a}^{\mu\alpha} \approx \mathring{\varepsilon}\mathbf{b}^{\mu\alpha}$ and

$$\widehat{\Gamma}^{\gamma}_{\ \alpha\beta} = \Gamma^{\gamma}_{\ \alpha\beta} + \mathring{\varepsilon}^2\, \mathring{z}^{\gamma}_{\ \alpha\beta}(\mathring{n}^a_i) + \ldots, \qquad {}_{\shortmid}\Gamma^{\gamma}_{\ \alpha\beta} = \widehat{\Gamma}^{\gamma}_{\ \alpha\beta} + \varepsilon\, {}_{\shortmid}z^{\gamma}_{\ \alpha\beta}(n^a_i)\ldots,$$
$${}^{s}R = {}^{s}\widehat{R} + \mathring{\varepsilon}^2\, \mathring{z}(g_{ij}, g_{ab}, n^a_i)\ldots, \qquad \mathbf{H}^2 = \mathring{\varepsilon}^2 \mathring{\mathbf{H}}(\mathbf{b}^{\mu\alpha}),$$
$${}^{s}\widehat{R} = {}_{\shortmid}R + \varepsilon\, {}_{\shortmid}z(g_{ij}, g_{ab}, n^a_i) + \ldots,$$

where

$$\Lambda \approx \mathring{\varepsilon}^2 \mathring{\Lambda}, \tag{49}$$

we transform $\mathcal{L}$ (43) into

$$\mathring{\mathcal{L}} = \sqrt{-g}\left[{}^{s}\widehat{R} + 2\mathring{\Lambda} - \frac{\mathring{\mathbf{H}}^2}{12} + \left(\frac{\mu^2}{4} + \beta\, {}^{s}\widehat{R}\right)\mathbf{b}^2 - \alpha\, \widehat{\mathbf{R}}_{\mu\nu}\mathbf{b}^{\mu\alpha}\mathbf{b}_{\alpha}^{\ \nu}\right] + \mathcal{O}(\mathbf{b}^3) \tag{50}$$

if and only if

$$\mathring{z}(g_{ij}, g_{ab}, n_i^a) = \gamma\, \widehat{\mathbf{R}}_{\mu\alpha\nu\beta}\mathbf{b}^{\mu\nu}\mathbf{b}^{\alpha\beta}. \tag{51}$$

The N-adapted variational field equations derived from (50) are

$$\frac{\mathbf{e}_\alpha(\sqrt{|\mathbf{g}_{\mu\nu}|}\mathring{\mathbf{H}}^{\alpha\beta\nu})}{\sqrt{|\mathbf{g}_{\mu\nu}|}} + (\mu^2 + 4\beta\, {}^{s}\widehat{R})\mathbf{b}^{\beta\nu} + 4\alpha\mathbf{b}^{\alpha(\nu}\widehat{\mathbf{R}}^{\beta)}_{\alpha} + \mathcal{O}(\mathbf{b}^2) = 0, \tag{52}$$

$$\widehat{\mathbf{R}}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\, {}^{s}\widehat{R} - \mathring{\varepsilon}^2\mathring{\Lambda}\mathbf{g}_{\mu\nu} + \mathcal{O}(\mathbf{b}^2) = 0, \tag{53}$$

where $\widehat{\mathbf{R}}^{\beta}_{\ \alpha\tau\nu}$, $\widehat{\mathbf{R}}_{\mu\nu}$ and ${}^{s}\widehat{R}$ are computed respectively by introducing the coefficients (39) into formulas (A.2) and (28). We can see that to the order $\mathcal{O}(\mathbf{b}^2)$ the fields equations decouple on the symmetric and nonsymmetric parts of d-metrics which allows us to consider a nonholonomic symmetric background defined by $(\mathbf{g}_{\mu\nu}, N_i^a, \widehat{\Gamma}^{\gamma}_{\ \alpha\beta})$ and to reduce the problem to the study of constrained dynamics of the antisymmetric d-field $\mathbf{a}^{\beta\nu}$ on this background.

4.2 Solutions with Ellipsoidal Symmetry

The simplest class of solutions for the system (52) and (53) can be constructed in the approximation that $\mathring{\varepsilon}^2\mathring{\Lambda} \sim 0$ and $\mu^2 \sim 0$.[9] For the ansatz

$$\mathring{\mathbf{H}}_{\alpha\beta\nu} = {}^{b}\lambda\sqrt{|\mathbf{g}_{\mu\nu}|}\epsilon_{\alpha\beta\nu}, \tag{54}$$

where ${}^{b}\lambda = \text{const}$ and $\epsilon_{\alpha\beta\nu}$ being the complete antisymmetric tensor, and any (vacuum) solution for

$$\widehat{\mathbf{R}}_{\mu\nu} = 0, \tag{55}$$

we generate decoupled solutions both for the symmetric and nonsymmetric part of metric. The nonsymmetric field $\mathbf{b}_{\beta\gamma}$ is any solution of

$${}^{b}\lambda\sqrt{|\mathbf{g}_{\mu\nu}|}\epsilon_{\alpha\beta\nu} = \mathbf{e}_\alpha\mathbf{b}_{\beta\gamma} + \mathbf{e}_\beta\mathbf{b}_{\gamma\alpha} + \mathbf{e}_\gamma\mathbf{b}_{\alpha\beta}, \tag{56}$$

which follows from formulas (44) and (54).

4.2.1 Anholonomic Deformations of the Schwarzschild Metric

Let us consider a primary quadratic element

$$\delta s^2_{[1]} = -d\xi^2 - r^2(\xi)\, d\vartheta^2 - r^2(\xi)\sin^2\vartheta\, d\varphi^2 + \varpi^2(\xi)\, dt^2, \tag{57}$$

[9] As a matter of principle, we can consider solutions with nonzero values of mass μ, but this will result in more sophisticate configurations for the nonsymmetric components of metrics which is not related to the problem of nonholonomic stabilization of noncommutative gravity theories, see Chap. 3 in [40], for similar details on constructing static black ellipsoid solutions in gravity with nonholonomic completely antisymmetric metric defined as a Proca field, and [43], for complex generalizations of such solutions to noncommutative gravity.

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$\check{g}_1 = -1, \qquad \check{g}_2 = -r^2(\xi), \qquad \check{h}_3 = -r^2(\xi)\sin^2\vartheta, \qquad \check{h}_4 = \varpi^2(\xi), \tag{58}$$

for $\xi = \int dr\ |1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2}|^{1/2}$ and $\varpi^2(r) = 1 - \frac{2m_0}{r} + \frac{\varepsilon}{r^2}$ and $x^1 = \xi, x^2 = \vartheta, y^3 = \varphi$, $y^4 = t$. For parameters $\varepsilon \to 0$ and m_0 being a point mass, the element (57) defines the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$. The parameter ε should not be confused with the square of the electric charge e^2 for the Reissner–Nordström metric. In our further considerations, we treat ε as a small parameter, for instance, defining a small deformation of a circle into an ellipse (eccentricity).

We construct a generic off-diagonal vacuum solution[10] by using nonholonomic deformations, $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$, where $(\check{g}_i, \check{h}_a)$ are given by data (58), when the new ansatz (target metric),

$$\begin{aligned} \delta s^2_{[\mathrm{def}]} &= -\eta_1(\xi)d\xi^2 - \eta_2(\xi)r^2(\xi)\ d\vartheta^2 \\ &\quad -\eta_3(\xi,\vartheta,\varphi)r^2(\xi)\sin^2\vartheta\ \delta\varphi^2 + \eta_4(\xi,\vartheta,\varphi)\varpi^2(\xi)\ \delta t^2, \\ \delta\varphi &= d\varphi + w_1(\xi,\vartheta,\varphi)d\xi + w_2(\xi,\vartheta,\varphi)d\vartheta, \\ \delta t &= dt + n_1(\xi,\vartheta)d\xi + n_2(\xi,\vartheta)d\vartheta, \end{aligned} \tag{59}$$

is supposed to solve (55). In formulas (59) there are used 3D spacial spherical coordinates, $(\xi(r), \vartheta, \varphi)$ or (r, ϑ, φ). The details on determining certain classes of coefficients for the target metric solving the vacuum Einstein equations for the canonical d-connection can be found in [2, 5, 41, 43] and Part II in [40]. Here we summarize the results which can be verified by direct computations:

The functions η_3 and η_4 can be generated by a function $b(\xi, \vartheta, \varphi)$ following conditions $-h_0^2(b^*)^2 = \eta_3(\xi,\vartheta,\varphi)r^2(\xi)\sin^2\vartheta$ and $b^2 = \eta_4(\xi,\vartheta,\varphi)\varpi^2(\xi)$, for

$$|\eta_3| = (h_0)^2|\check{h}_4/\check{h}_3|\left[\left(\sqrt{|\eta_4|}\right)^*\right]^2, \tag{60}$$

with $h_0 = \mathrm{const}$, where $\check{h}_a$ are stated by the Schwarzschild solution for the chosen system of coordinates and η_4 can be any function satisfying the condition $\eta_4^* = \partial\eta_4/\partial\varphi \neq 0$. We can compute the polarizations η_1 and η_2, when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi,\vartheta)}$ with ψ solving $\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\vartheta^2} = 0$. The nontrivial values of N-connection coefficients $N_i^3 = w_i(\xi,\vartheta,\varphi)$ and $N_i^4 = n_i(\xi,\vartheta,\varphi)$, when $i = 1, 2$, for vacuum configurations with the Levi–Civita connection ∇ are given by

$$w_1 = \partial_\xi(\sqrt{|\eta_4|}\varpi)/\left(\sqrt{|\eta_4|}\right)^*\varpi, \qquad w_2 = \partial_\vartheta(\sqrt{|\eta_4|})/\left(\sqrt{|\eta_4|}\right)^*$$

and any $n_{1,2} = {}^1n_{1,2}(\xi,\vartheta)$ for which $\partial_\vartheta({}^1n_1) - \partial_\xi({}^1n_2) = 0$, when, for instance $\partial_\xi = \partial/\partial\xi$. In a more general case, when $\nabla \neq \widehat{\mathbf{D}}$, but the nonholonomic vacuum equation (55) is solved, we have to take

$$n_{1,2}(\xi,\vartheta,\varphi) = {}^1n_{1,2}(\xi,\vartheta) + {}^2n_{1,2}(\xi,\vartheta)\int d\varphi\ h_3/\left(\sqrt{|h_4|}\right)^3,$$

[10] It can not diagonalized by coordinate transforms.

for ${}^{1}n_{1,2}(\xi,\vartheta)$ and ${}^{2}n_{1,2}(\xi,\vartheta)$ being certain integration functions to be defined from certain boundary conditions, or constrained additionally to solve certain compatibility equations in some limits.

Putting the defined values of the coefficients in the ansatz (59), we construct a class of exact vacuum solutions of the Einstein equations for the canonical d-connection (in particular, for the Levi–Civita connection) defining stationary nonholonomic deformations of the Schwarzschild metric,

$$\delta s^2_{[1]} = -e^{\psi}\left(d\xi^2 + d\vartheta^2\right) - h_0^2\left[\left(\sqrt{|\eta_4|}\right)^*\right]^2 \varpi^2\,\delta\varphi^2 + \eta_4\varpi^2\,\delta t^2,$$
$$\delta\varphi = d\varphi + \frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{(\sqrt{|\eta_4|})^*\varpi}d\xi + \frac{\partial_\vartheta(\sqrt{|\eta_4|})}{(\sqrt{|\eta_4|})^*}d\vartheta, \qquad \delta t = dt + n_1 d\xi + n_2 d\vartheta. \tag{61}$$

Such solutions were constructed to define anholonomic transform of a static black hole solution into stationary vacuum Einstein (non)holonomic spaces with local anisotropy (on coordinate φ) defined by an arbitrary function $\eta_4(\xi,\vartheta,\varphi)$ with $\partial_\varphi\eta_4 \neq 0$, an arbitrary $\psi(\xi,\vartheta)$ solving the 2D Laplace equation and certain integration functions ${}^{1}n_{1,2}(\xi,\vartheta)$ and integration constant h_0^2. In general, the solutions from the target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions. We can also consider a prescribed physical situation when, for instance, η_4 mimics 3D, or 2D, solitonic polarizations on coordinates ξ,ϑ,φ, or on ξ,φ, see [2, 5, 43].

4.2.2 Solutions with Small Nonholonomic Polarizations

The class of solutions (65) is defined in a very general form. Let us extract a subclasses of solutions related to the Schwarzschild metric. We consider decompositions on a small parameter $0<\varepsilon<1$ in (61), when

$$\sqrt{|\eta_3|} = q_3^{\hat 0}(\xi,\vartheta,\varphi) + \varepsilon q_3^{\hat 1}(\xi,\vartheta,\varphi) + \varepsilon^2 q_3^{\hat 2}(\xi,\vartheta,\varphi)\ldots,$$
$$\sqrt{|\eta_4|} = 1 + \varepsilon q_4^{\hat 1}(\xi,\vartheta,\varphi) + \varepsilon^2 q_4^{\hat 2}(\xi,\vartheta,\varphi)\ldots,$$

where the "hat" indices label the coefficients multiplied to $\varepsilon,\varepsilon^2,\ldots$. The conditions (60) are expressed: $\varepsilon h_0\sqrt{|\frac{\check h_4}{\check h_3}|}(q_4^{\hat 1})^* = q_3^{\hat 0},\ \varepsilon^2 h_0\sqrt{|\frac{\check h_4}{\check h_3}|}\,(q_4^{\hat 2})^* = \varepsilon q_3^{\hat 1},\ldots$. We take the integration constant, for instance, to satisfy the condition $\varepsilon h_0 = 1$ (choosing a corresponding distributions and system of coordinates). This condition will be important in order to get stable solutions for certain $\varepsilon \neq 0$, but small, i.e. $0<\varepsilon<1$. For such small deformations, we prescribe a function $q_3^{\hat 0}$ and define $q_4^{\hat 1}$, integrating on φ (or inversely, prescribing $q_4^{\hat 1}$, then taking the partial derivative ∂_φ, to compute $q_3^{\hat 0}$). In a similar form, there are related the coefficients $q_3^{\hat 1}$ and $q_3^{\hat 2}$. An important physical situation arises when we select the conditions when such small nonholonomic deformations define rotoid configurations. This is possible, for instance, if

$$2q_4^{\hat 1} = \frac{q_0(r)}{4m_0^2}\sin(\omega_0\varphi+\varphi_0) - \frac{1}{r^2}, \tag{62}$$

where ω_0 and φ_0 are constants and the function $q_0(r)$ has to be defined by fixing certain boundary conditions for polarizations. In this case, the coefficient before δt^2 is

$$\eta_4 \varpi^2 = 1 - \frac{2m_0}{r} + \varepsilon \left(\frac{1}{r^2} + 2q_4^{\hat{1}} \right). \tag{63}$$

This coefficient vanishes and defines a small deformation of the Schwarzschild spherical horizon into a an ellipsoidal one (rotoid configuration) given by $r_+ \simeq \frac{2\mu}{1+\varepsilon \frac{q_0(r)}{4m_0^2} \sin(\omega_0\varphi+\varphi_0)}$. Such solutions with ellipsoid symmetry seem to define static black ellipsoids which are stable (they were investigated in details in [41, 42]). The ellipsoid configurations were proven to be stable under perturbations and transform into the Schwarzschild solution far away from the ellipsoidal horizon. In general relativity, this class of vacuum metrics violates the conditions of black hole uniqueness theorems [47] because the "surface" gravity is not constant for stationary black ellipsoid deformations.

We can construct an infinite number of ellipsoidal locally anisotropic black hole deformations. Nevertheless, they present physical interest because they preserve the spherical topology, have the Minkowski asymptotic and the deformations can be associated to certain classes of geometric spacetime distorsions related to generic off-diagonal metric terms. Putting $\varphi_0 = 0$, in the limit $\omega_0 \to 0$, we get $q_4^{\hat{1}} \to 0$ in (62). To get a smooth limit to the Schwarzschild solution we have to state the limit $q_3^{\hat{0}} \to 1$ for $\varepsilon \to 0$.

Let us summarize the above presented approximations for ellipsoidal symmetries: For (63), we have

$$h_4 = \eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi) = 1 - \frac{2m_0}{r} + \varepsilon \frac{q_0(r)}{4m_0} \sin(\omega_0\varphi + \varphi_0) + \mathcal{O}(\varepsilon^2),$$

$$h_3 = \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2\vartheta = h_0^2[(\sqrt{|\eta_4|})^*]^2 \varpi^2(\xi) = (\varepsilon h_0)^2[(q_4^{\hat{1}})^*]^2,$$

which results in $h_3 = (\varepsilon h_0)^2 \frac{q_0(r)\omega_0^2}{16m_0} \cos^2(\omega_0\varphi + \varphi_0) + \mathcal{O}(\varepsilon^3)$, where we must preserve the second order on ε^2 if $\varepsilon h_0 \sim 1$. To get a smooth limit of off-diagonal coefficients in solutions to the Schwarzschild metric (57), we state that after integrations one approximates the N-connection coefficients as $N_i^a \sim \varepsilon n_i^a$. Putting together all decompositions of coefficients on ε in (61), we get a family of ellipsoidal solution of equations (55) decomposed on eccentricity ε,

$$\begin{aligned} \delta s_{[1]}^2 &= -e^{\varepsilon\psi}\left(d\xi^2 + d\vartheta^2\right) - (\varepsilon h_0)^2 \frac{q_0(r)\omega_0^2}{16m_0} \cos^2(\omega_0\varphi + \varphi_0)\delta\varphi^2 \\ &\quad + \left[1 - \frac{2m_0}{r} + \varepsilon \frac{q_0(r)}{4m_0} \sin(\omega_0\varphi + \varphi_0) + \mathcal{O}(\varepsilon^2)\right]\delta t^2, \\ \delta\varphi &= d\varphi + \varepsilon \frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{(\sqrt{|\eta_4|})^*\varpi} d\xi + \varepsilon \frac{\partial_\vartheta(\sqrt{|\eta_4|})}{(\sqrt{|\eta_4|})^*} d\vartheta, \qquad \delta t = dt + \varepsilon n_1 d\xi + \varepsilon n_2 d\vartheta. \end{aligned} \tag{64}$$

One can be defined certain more special cases when $q_4^{\hat{2}}$ and $q_3^{\hat{1}}$ (as a consequence) are of solitonic locally anisotropic nature. In result, such solutions will define small stationary deformations of the Schwarzschild solution embedded into a background polarized by anisotropic solitonic waves.

Now, we show how we can solve the problem of stability related to the condition (51): Let us consider a small cosmological constant of type (49) stated only in the horizontal spacetime distribution ${}^{h}\Lambda \approx \mathring{\varepsilon}^{2}\ {}^{h}\mathring{\Lambda}$, but ${}^{v}\Lambda = 0$.[11] By straightforward computations, we can verify that the symmetric part of the ansatz

$$\begin{aligned}
\delta s^2_{[1]} &= -e^{\varepsilon\psi + \mathring{\varepsilon}^2 \mathring{\psi}}\left(d\xi^2 + d\vartheta^2\right) - (\varepsilon h_0)^2 \frac{q_0(r)\omega_0^2}{16 m_0}\cos^2(\omega_0\varphi + \varphi_0)\,\delta\varphi^2 \\
&\quad + \left[1 - \frac{2m_0}{r} + \varepsilon\frac{q_0(r)}{4m_0}\sin(\omega_0\varphi + \varphi_0) + \mathcal{O}(\varepsilon^2)\right]\delta t^2 + \mathring{\varepsilon}\mathbf{b}_{\alpha\beta}\,\mathbf{e}^{\alpha}\wedge\mathbf{e}^{\beta}, \\
\delta\varphi &= d\varphi + \varepsilon\frac{\partial_\xi(\sqrt{|\eta_4|}\varpi)}{(\sqrt{|\eta_4|})^{*}\varpi}d\xi + \varepsilon\frac{\partial_\vartheta(\sqrt{|\eta_4|})}{(\sqrt{|\eta_4|})^{*}}d\vartheta, \\
\delta t &= dt + \varepsilon n_1 d\xi + \varepsilon n_2 d\vartheta.
\end{aligned} \tag{65}$$

solves the equations

$$\begin{aligned}
R_{ij} &= \widehat{R}_{ij} + \mathring{\varepsilon}^2\mathring{z}_{ij}, \quad \text{for } \mathring{z}_{ij} = -\,{}^{h}\mathring{\Lambda}e^{\varepsilon\psi + \mathring{\varepsilon}^2\mathring{\psi}}\delta_{ij},\ \widehat{R}_{ij} = 0 \\
R_{ia} &= \widehat{R}_{ia} = 0, \qquad R_{ai} = \widehat{R}_{ai} = 0, \qquad R_{ab} = \widehat{R}_{ab} = 0,
\end{aligned} \tag{66}$$

where $\mathbf{e}^{\alpha} = (d\xi, d\vartheta, \delta\varphi, \delta t)$, $\mathring{\psi}$ is the solution of $\frac{\partial^2\mathring{\psi}}{\partial\xi^2} + \frac{\partial^2\mathring{\psi}}{\partial\vartheta^2} = {}^{h}\mathring{\Lambda}$. If the nonsymmetric part of (65) is with $\mathbf{b}_{\alpha\beta}$ being a solution of (56), the rest of coefficients are constrained to satisfy above mentioned conditions, we generate a class of both nonholonomic and nonsymmetric metric deformations of the Schwarzschild metric which defines a family of two parametric nonholonomic solutions in noncommutative gravity theories (when the gravitational field equations are approximated by (52) and (52)). The stability conditions (51) result in $\mathring{z} = g^{ij}\mathring{z}_{ij}(g_{ij,}g_{ab},, n_i^a) = {}^{h}\mathring{\Lambda} = \gamma\ \widehat{\mathbf{R}}_{\mu\alpha\nu\beta}\mathbf{b}^{\mu\nu}\mathbf{b}^{\alpha\beta}$. This imposes a constraint of the generating function $q_4^{\hat{1}}(\xi, \vartheta, \varphi)$ and integration functions and constants, of type $q_0(r)$ and ${}^{1}n_i(\xi, \vartheta)$ and ${}^{2}n_i(\xi, \vartheta)$, which selects of subspace in the integral variety of solutions of (66). We have $\mathring{z} = 0$ and $\widehat{\mathbf{R}}_{\mu\alpha\nu\beta}$, for any nonzero γ, in the case of teleparallel nonholonomic manifolds, see Chap. 1 in [40] (we note that for such configurations the Riemann curvature for the Levi–Civita connection, in general, is not zero).

We conclude that the presence of a small cosmological constant ${}^{h}\Lambda \approx \mathring{\varepsilon}^{2}\ {}^{h}\mathring{\Lambda}$ may stabilize additionally the solutions but stability can be obtained also for vanishing cosmological constants. Constructing such solutions we considered, for simplicity, that the mass of effective gauge fields is very small. In a more general case, we can generate nonsymmetric metrics with effective Proca fields with nonzero mass and nonzero cosmological constants, see more sophisticate constructions in [39, 40, 43].

5 Conclusions and Discussion

In this article we developed a new method of stabilization in nonsymmetric gravity theories and spacetimes provided with nonholonomic distributions and canonically induced anholonomic frames with associated nonlinear connection (N-connection) structures. For general effective Lagrangians modelling nonsymmetric gravity theories on (non) holonomic backgrounds, we shown how to construct stable and nonstable solutions. We argued that the

[11] The techniques presented in [2, 5, 40, 41] allows us to construct solutions for nontrivial values ${}^{v}\Lambda$, but this would result in modifications of the formulas for the vertical part of d-metric and N-connection coefficients, which is related to a more cumbersome calculus; in this work, we analyze the simplest examples.

corresponding systems of field equations possess different types of gauge like and nonholonomically deformed symmetries which may stabilize, or inversely, evolve into instabilities which depends on the type of imposed constrains and ansatz for the symmetric and nonsymmetric components of metric and related N-connection and linear connection structures.

The N-connection geometry and the formalism of parametric nonholonomic frame transforms are the key prerequisites of the so-called anholonomic frame method of constructing exact and approximate solutions in Einstein gravity and various generalizations to (non)symmetric metrics, metric-affine, noncommutative, string like and Lagrange–Finsler gravity models, see reviews and explicit examples in [5, 39, 40, 43]. Such geometric methods allow us to generate very general classes of solutions of nonlinear field and constraints equations, depending on three and four variables and on infinite number of parameters, and solve certain stability problems in various models of gravity. For simplicity, in this paper we consider the nonholonomic stabilization method for a class of solutions with ellipsoidal symmetries which transform into the Schwarzschild background for small eccentricities and small nonsymmetry (of metrics) parameters.

The idea to use Lagrange multipliers and dynamical constraints proposed and elaborated in [38], in order to solve instabilities discovered in nonsymmetric gravity theory by Clayton [35, 36], contains already a strong connection to the nonholonomic geometry and field dynamics. This work develops that dynamical constraint direction to the case of nonholonomic parametric deformations following certain results from [1, 2] (on the geometry of generalized spaces and Ricci flows constrained to result in nonholonomic and (non)symmetric structures). This way we can solve the Janssen–Prokopec stability problem in nonsymmetric gravity theory [31–33] and develop a new (nonholonomic) direction in (non) symmetric gravity and related spacetime geometry. Here we also note that nonsymmetric components of metrics arise naturally as generalized almost symplectic structures in deformation quantization of gravity [4, 13] when corresponding almost Kähler models are elaborated for quantum models. It was proved how general relativity can be represented equivalently in nonsymmetric almost symplectic variables for a canonical model on a corresponding almost Kähler spaces. For such a model of "nonsymmetric" gravity/general relativity, the questions on stability of solutions is to be analyzed as in general relativity, together with additional considerations for nonholonomic constraints.

Following the above-mentioned results, we have to conclude that nonsymmetric metrics and connections are defined naturally from very general constructions in modern geometry, nonlinear functional analysis and theoretical methods in gravity and particle physics. Such nonsymmetric generalizations of classical and quantum gravity models can not be prohibited by some examples when a gauge symmetry or stability scenaria fail to be obtained for a fixed flat or curved background like in [31–36]. It is almost sure that certain nonlinear mathematical techniques always can be provided in order to construct stable, or un-stable, solutions, with evolutions of necessary type; as well one can be elaborated well defined physical scenaria and alternatives. This is typical for generic nonlinear theories like general relativity and nonsymmetric gravity theories.

Of course, there exists the so-called generality problem in nonsymmetric gravity theories when a guiding principle has to be formulated in order to select from nine and more constants and extra terms in generalized Lagrangians (at least by 11 undetermined parameters come from the full theory and the decomposition of the metric tensor). It may be that (non)symmetric corrections to metrics and connections can be derived following certain geometric principles in Ricci flow and/or deformation quantization theories, not only from the variational principle for generalized field interactions and imposed nonholonomic constraints. One also has to be exploited intensively certain variants of selection from different theories following existing and further experimental data like in [25–27, 33], see also references therein. At this moment, there are none theoretical and experimental prohibitions for nonsymmetric metrics which would be established in modern cosmology, astrophysics and experimental particle physics.

Acknowledgements The work was performed and discussed during some visits at Fields and Perimeter Institutes. The author is grateful to referees for hard work and suggested important modifications.

Appendix: Some Component Formulas and Redefinition of Constants

The N-adapted coefficients of torsion is computed in the form

$$
\begin{aligned}
&\mathcal{T}^{\alpha} \doteqdot \mathbf{D}\mathbf{e}^{\alpha} = d\mathbf{e}^{\alpha} + \Gamma^{\alpha}_{\ \beta} \wedge \mathbf{e}^{\beta}, \\
\text{for}\quad & T^{i}_{\ jk} = L^{i}_{\ jk} - L^{i}_{\ kj}, \qquad T^{i}_{\ ja} = C^{i}_{\ ja}, \qquad T^{a}_{\ ji} = \Omega^{a}_{\ ji}, \\
& T^{a}_{\ bi} = \frac{\partial N^{a}_{i}}{\partial y^{b}} - L^{a}_{\ bi}, \qquad T^{a}_{\ bc} = C^{a}_{\ bc} - C^{a}_{\ cb},
\end{aligned}
\tag{A.1}
$$

where $\Omega^{a}_{\ ji}$ is the curvature of N-connection (13). We cite articles [1, 5] for reviews of results and more details.

By a straightforward d-form calculus, we can find the N-adapted components of the curvature of a d-connection $\mathbf{D}$,

$$
\mathcal{R}^{\alpha}_{\ \beta} \doteqdot \mathbf{D}\Gamma^{\alpha}_{\ \beta} = d\Gamma^{\alpha}_{\ \beta} - \Gamma^{\gamma}_{\ \beta} \wedge \Gamma^{\alpha}_{\ \gamma} = \mathbf{R}^{\alpha}_{\ \beta\gamma\delta}\mathbf{e}^{\gamma} \wedge \mathbf{e}^{\delta},
$$

i.e. the d-curvature,

$$
\begin{aligned}
R^{i}_{\ hjk} &= \mathbf{e}_{k}\left(L^{i}_{\ hj}\right) - \mathbf{e}_{j}\left(L^{i}_{\ hk}\right) + L^{m}_{\ hj}L^{i}_{\ mk} - L^{m}_{\ hk}L^{i}_{\ mj} - C^{i}_{\ ha}\Omega^{a}_{\ kj}, \\
R^{a}_{\ bjk} &= \mathbf{e}_{k}\left(L^{a}_{\ bj}\right) - \mathbf{e}_{j}\left(L^{a}_{\ bk}\right) + L^{c}_{\ bj}L^{a}_{\ ck} - L^{c}_{\ bk}L^{a}_{\ cj} - C^{a}_{\ bc}\Omega^{c}_{\ kj}, \\
R^{i}_{\ jka} &= e_{a}L^{i}_{\ jk} - \mathbf{D}_{k}C^{i}_{\ ja} + C^{i}_{\ jb}T^{b}_{\ ka}, \\
R^{c}_{\ bka} &= e_{a}L^{c}_{\ bk} - \mathbf{D}_{k}C^{c}_{\ ba} + C^{c}_{\ bd}T^{c}_{\ ka}, \\
R^{i}_{\ jbc} &= e_{c}C^{i}_{\ jb} - e_{b}C^{i}_{\ jc} + C^{h}_{\ jb}C^{i}_{\ hc} - C^{h}_{\ jc}C^{i}_{\ hb}, \\
R^{a}_{\ bcd} &= e_{d}C^{a}_{\ bc} - e_{c}C^{a}_{\ bd} + C^{e}_{\ bc}C^{a}_{\ ed} - C^{e}_{\ bd}C^{a}_{\ ec}.
\end{aligned}
\tag{A.2}
$$

Contracting the first and forth indices $\mathbf{R}_{\beta\gamma} = \mathbf{R}^{\alpha}_{\ \beta\gamma\alpha}$, one gets the N-adapted coefficients for the Ricci tensor $\mathcal{R}ic \doteqdot \{\mathbf{R}_{\beta\gamma} = (R_{ij}, R_{ia}, R_{ai}, R_{ab})\}$. See explicit formulas in [39]. It should be noted here that for general d-connections the Ricci tensor is not symmetric, i.e. $\mathbf{R}_{\beta\gamma} \neq \mathbf{R}_{\gamma\beta}$.

Finally, we note that there are two scalar curvatures, ${}^{s}R$ and ${}^{s}\check{R}$, of a d-connection defined by formulas ${}^{s}R = \mathbf{g}^{\beta\gamma}\mathbf{R}_{\beta\gamma}$ and ${}^{s}\check{R} = \check{\mathbf{g}}^{\beta\gamma}\mathbf{R}_{\beta\gamma}$. Both geometric objects can be considered in generalized gravity theories.

Similar formulas holds true, for instance, for the Levi–Civita linear connection $\bigtriangledown = \{{}_{\shortmid}\Gamma^{\alpha}_{\beta\gamma}\}$ is uniquely defined by the symmetric metric structure (34) by the conditions ${}_{\shortmid}\mathcal{T} = 0$ and $\bigtriangledown\mathbf{g} = 0$. It should be noted that this connection is not adapted to the distribution (8) because it does not preserve under parallelism the h- and v-distribution. Any geometric construction for the canonical d-connection, $\widehat{\mathbf{D}} = \{{}_{\shortmid}\Gamma^{\gamma}_{\ \alpha\beta}\}$, can be re-defined for the Levi–Civita connection, $\widehat{\Gamma}^{\gamma}{}_{\alpha\beta}$, using formula ${}_{\shortmid}\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\Gamma}^{\gamma}{}_{\alpha\beta} + {}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}$, where distorsion tensor ${}_{\shortmid}Z^{\gamma}{}_{\alpha\beta}$ is defined by N-adapted coefficients

$$\begin{aligned}
&{}_{\shortmid}Z^{i}_{jk} = 0, \qquad {}_{\shortmid}Z^{a}_{jk} = -C^{i}_{jb} g_{ik} h^{ab} - \frac{1}{2}\Omega^{a}_{jk}, \qquad {}_{\shortmid}Z^{i}_{bk} = \frac{1}{2}\Omega^{c}_{jk} h_{cb} g^{ji} - \Xi^{ih}_{jk}\, C^{j}_{hb}, \\
&{}_{\shortmid}Z^{a}_{bk} = {}^{+}\Xi^{ab}_{cd}\, {}^{\circ}L^{c}_{bk}, \qquad {}_{\shortmid}Z^{i}_{kb} = \frac{1}{2}\Omega^{a}_{jk} h_{cb} g^{ji} + \Xi^{ih}_{jk}\, C^{j}_{hb}, \\
&{}_{\shortmid}Z^{a}_{jb} = -\,{}^{-}\Xi^{ad}_{cb}\; {}^{\circ}L^{c}_{dj}, \qquad {}_{\shortmid}Z^{a}_{bc} = 0, \qquad {}_{\shortmid}Z^{i}_{ab} = -\frac{g^{ij}}{2}\left[{}^{\circ}L^{c}_{aj} h_{cb} + {}^{\circ}L^{c}_{bj} h_{ca}\right], \\
&\Xi^{ih}_{jk} = \frac{1}{2}(\delta^{i}_{j}\delta^{h}_{k} - g_{jk} g^{ih}), \qquad {}^{\pm}\Xi^{ab}_{cd} = \frac{1}{2}(\delta^{a}_{c}\delta^{b}_{d} + h_{cd} h^{ab}),
\end{aligned} \tag{A.3}$$

for ${}^{\circ}L^{c}_{aj} = L^{c}_{aj} - e_a(N^{c}_{j})$. Both d-connections and d-tensor are determined by the generic off-diagonal metric (34), or (equivalently) by d-metric (33) and the coefficients of N-connection (9) [39]. If we work with nonholonomic constraints on the dynamics/geometry of gravity fields, it is more convenient to use a N-adapted approach. For other purposes, it is preferred to use only the Levi–Civita connection.

In order to get a convenient form of effective Lagrangian, the constants from (40) and (42) are re-defined in the form:

$$\begin{aligned}
&\alpha = \rho + \Xi - 1,\ \beta = \frac{1}{2}\left(\frac{1}{2} - \rho + 2\sigma\right), \\
&\gamma = \Xi = 3\Sigma^2\theta^2\,[2(c_1+c_3) + 1 - b_2] + \Sigma^2\left[d - \frac{b_2}{3} + \frac{2}{3}(c_1+c_3) - \frac{3}{8}\mathbb{L}^2\right] \\
&\qquad + \Sigma\{2\left[\theta(1+b_1) - \frac{2}{3} - \frac{8a_1}{3} + \frac{a_2}{2} - \frac{b_1}{3} - \left(a_1 + \frac{1}{2}\right)\frac{\mathbb{L}^2}{2}\right] \\
&\qquad - \theta(\phi - \xi)\,[4(c_1+c_3) - 2b_2 + 2]\}
\end{aligned} \tag{A.4}$$

for

$$\begin{aligned}
&\Sigma = \frac{3a_2/2 - b_1 - 8a_1 - 2}{b_2 - 2(c_1+c_3) - 3d_1}, \qquad \mathbb{L} = \frac{2}{3}\frac{(1+2a_1)[b_2 - 3d_1 - 2(c_1+c_3)]}{2 + 8a_1 - \frac{3}{2}a_2 - b_1} \\
&\mathbb{K} = \frac{3(a_1 - a_2/4)}{(1+2a_1)}\mathbb{L} + d_1 - \frac{b_2}{3} + \frac{2}{3}(c_1+c_3), \qquad \theta \equiv \frac{2\mathbb{K} + \mathbb{L}}{\mathcal{A} - \mathcal{B}}, \\
&\xi \equiv \frac{(\mathcal{A} + 3\mathcal{B})(b_1+1)}{\mathcal{A}^2 + \mathcal{A}\mathcal{B} - \mathcal{B}^2}, \qquad \phi = \psi = \frac{(\mathcal{A} + \mathcal{B})(b_1+1)}{\mathcal{A}^2 + \mathcal{A}\mathcal{B} - \mathcal{B}^2}, \\
&\mathcal{A} = 2(1 - b_2 + c_1 + c_2), \qquad \mathcal{B} = -2(c_1+c_3), \\
&\Psi = (b_2 - 1)(\xi - \phi)^2 + 2\phi(1+b_1), \qquad \Phi = \frac{1}{3}(\phi^2 + 2\xi\phi)(b_2 + c_1 + c_3 - 1)^2, \\
&\Omega = (c_1+c_3)(\xi - \phi)^2 + \xi(1+b_1),
\end{aligned}$$

where the conditions

$$\Omega + 3\Phi = -\frac{1}{4} \quad \text{and} \quad \Xi = \Psi - 2\Omega \tag{A.5}$$

have to be imposed in order to get a stable effective Lagrangian in the flat space limit.

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